MAT 102 SPRING 2008
FEEDBACK ON HOMEWORK 1

1. Some common mistakes.

- **Q30 of §4.1.** One should be aware that $f(t) = |t - 5|$ is not differentiable at $t = 5$. That means the function is a sharp pike there (in particular, $f'(5)$ is NOT 0). Hence 5 is a critical point for the function, and one needs to consider that point when one locates the global maximum and minimum of the function. See below for the full solution.
- **Q65a of §4.1.** Many of you just computed the critical point of C and claim without giving a reason that C must attain a minimum there. This is not really immediate. Presumably, when you know $8/\sqrt{5} \simeq 3.58$ is a critical point of C, you only know that it may be a local maximum, a local minimum, or worse, neither. One actually needs to compute the values of C at the end-points of the domain that we are concerned with, namely at 0 and 9, to see that the critical point is a minimum of C (and not a maximum for instance). See below for the full solution.

  Indeed the same problem occurs in almost every other optimization problem. It's important to understand that you will still have to evaluate the function at the end-points to find global maxima and minima, even though you are working on a word problem.
- **Q48 of §4.1.** This is a hard problem. Many of you found the critical points of the function, but few could explain why they are indeed global maxima and minima. The proper reason is the following: first, notice that the critical points of y are $(-2, -1/2)$ and $(0, 1/2)$. Next, it is easily verified that
$$\lim_{x \to \pm\infty} y(x) = \lim_{x \to \pm\infty} \frac{x+1}{x^2+2x+2} = 0.$$
  This says $y(x)$ is very close to 0 as long as x is close to infinity. Hence roughly speaking, the maxima and minima cannot be attained at infinity, and we have $(2, -1/2)$ being a global minimum, and $(0, 1/2)$ being a global maximum. More precise arguments can be given, but we will leave that to you if you are interested. See below for the full solution.
- **Q40 of §4.3.** This is a bit tricky. $x = -1$ is a critical point of $k(x)$, but it is neither a local maximum nor a local minimum; indeed the function is increasing on the whole interval $(-\infty, 0]$, because $k'(x) = 3(x+1)^2 \geq 0$ for any x. It also follows from this that the end-point, $(0, 1)$, is the (local and global) maximum of k on $(-\infty, 0]$. See below for full solution.
- **Plotting graphs.** In plotting a graph, make sure you study carefully where the function is increasing and decreasing, and where the function is concave up/down. Give reasons as you proceed - many of you simply jump ahead and draw the graph. I guess you may be running out of time, since this was a pretty long homework. Just try to avoid that next time.
- **Some other generalities.** In homeworks, quizzes and exams, you will be expected to provide reasons for each step of your calculation if you are to receive full credit. You should also write in a way that an ordinary mathematics student would be able to follow what you are doing without having

to think exceedingly hard. This is indeed a good training in presenting and communicating logical arguments. In doing so you are also acting in favour of yourself: if by any chance your answer is not correct, we look to your intermediate steps and assign partial credits based on the reasoning you provide. See below for examples.

2. Solutions.

Just to help you lift off in the course, and see how one could present an argument more logically, I have prepared the following solutions to the exercises, and I hope you will find them helpful.

- **Q30 of §4.1.** Let $f(t) = |t-5|$ for $4 \leq t \leq 7$. Then
$$f(t) = \begin{cases} t-5 & \text{if } t \geq 5 \\ -(t-5) & \text{if } t < 5 \end{cases}.$$
It is a piecewise defined function, and we can compute its derivative piecewise, taking care of the points of division ($t = 5$ in this case) separately:
$$f'(t) = \begin{cases} 1 & \text{if } t > 5 \\ \textbf{undefined} & \text{if } t = 5 \\ -1 & \text{if } t < 5 \end{cases}.$$
Hence the only critical point in the open interval $4 < t < 7$ is $t = 5$.
To compute the absolute maximum, we compute the function values at the critical points, as well as the end-points of the domain of definition.
$$f(4) = 1, \quad f(7) = 2, \quad f(5) = 0.$$
Since $f(7) > f(4)$ and $f(7) > f(5)$, the absolute maximum cannot occur at $t = 4$ or 5. Hence the absolute maximum is $(7, 2)$, and similarly the absolute minimum is $(5, 0)$.
- **Q48 of §4.1.** Let $y(x) = \frac{x+1}{x^2+2x+2}$. Then
$$y'(x) = \frac{-x(x+2)}{(x^2+2x+2)^2}.$$
It is defined everywhere because the denominator
$$(x^2+2x+2)^2 = ((x+1)^2+1)^2 \geq 1$$
for any real number x. Hence setting $y'(x) = 0$, i.e. $x(x+2) = 0$, we see that the only critical points of y are at $x = 0$ or $x = -2$, i.e. $(0, 1/2)$ and $(-2, -1/2)$.
Next
$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 + \frac{2}{x} + \frac{2}{x^2}} = 0.$$
Hence $y(x)$ is very close to zero when x is very close to (positive or negative) infinity; in particular the function cannot for instance grow at infinity. As a result, $y(x)$ has an absolute maximum, and it occurs at the point $(0, 1/2)$. Similarly y has an absolute minimum, and it occurs at $(-2, -1/2)$.
- **Q65a of §4.1.** Let x be the distance from A to B. Then the cost of building the pipe is
$$C(x) = 300000(16 + x^2)^{1/2} + 200000(9 - x), \quad 0 \leq x \leq 9.$$

Hence

$$\begin{aligned} C'(x) &= 300000\left(\frac{1}{2}(16+x^2)^{-1/2}\right)(2x) - 200000 \\ &= \frac{300000x}{\sqrt{16+x^2}} - 200000. \end{aligned}$$

$C'(x)$ is everywhere defined, and $C'(x) = 0$ precisely when

$$\begin{aligned} \frac{3x}{\sqrt{16+x^2}} &= 2, \\ \text{i.e. } 9x^2 &= 4(16+x^2). \end{aligned}$$

Solving, $x = \frac{8}{\sqrt{5}}$ or $-\frac{8}{\sqrt{5}}$ (rejected). Hence $x = \frac{8}{\sqrt{5}}$ is the only critical point of C in the domain of definition.

To find the absolute minimum, we compute

$$C(0) = 3000000, \quad C(9) = 300000\sqrt{97} \simeq 2900000, \quad C\left(\frac{8}{\sqrt{5}}\right) \simeq 2700000.$$

Hence the minimum value of C occur at $x = \frac{8}{\sqrt{5}}$, and the point B should be $\frac{8}{\sqrt{5}}$ mi from A.

(Alternatively, observe that $C'(x) < 0$ when $0 < x < \frac{8}{\sqrt{5}}$, and $C'(y) > 0$ when $\frac{8}{\sqrt{5}} < x < 9$. Hence $C(x)$ achieves the minimum at $x = \frac{8}{\sqrt{5}}$.)

- **Q40 of §4.3.** Let $k(x) = x^3 + 3x^2 + 3x + 1$ for $-\infty < x \leq 0$. Then $k'(x) = 3x^2 + 6x + 3 = 3(x+1)^2$ for any real number x. Setting $k'(x) = 0$, we get $x = -1$. This is the only critical point of the function. Let's analyze the behaviour of $k(x)$.

x	$-\infty < x < -1$	-1	$-1 < x < 0$
$(x+1)$	$-$ve	0	$+$ve
$(x+1)$	$-$ve	0	$+$ve
$k'(x) = 3(x+1)^2$	$+$ve	0	$+$ve
$k(x)$	increasing		increasing

As a result, -1 is neither an local maximum nor a local minimum of $k(x)$, and $k(x)$ has an absolute maximum on the interval $-\infty < x \leq 0$, which is achieved when $x = 0$ at the point $(0, 1)$.

(The tricky part here is that $k'(x)$ does not change sign as you move across the critical point -1. It helps to **always factorize** $k'(x)$ when you want to locate critical points and determine where it is increasing.)

3. Problems in presentation.

- **Q23 of §4.1.** Do **not** write

$$g'(x) = \frac{x}{\sqrt{4-x^2}} = 0, \quad x = 0.$$

This is an unclear sentence by itself: better say for instance

$$g'(x) = \frac{x}{\sqrt{4-x^2}} \text{ for any } -2 < x < 2, \text{ so setting } g'(x) = 0 \text{ we get } x = 0.$$

Similarly, **don't** write

$$g'(x) = \frac{x}{\sqrt{4-x^2}} = 0, 2, -2 \text{ critical point.}$$

- When you want to write down a point, make sure you embrace them with a bracket. In other words, $(0, 2)$ is a point, while $0, 2$ possibly means '0 or 2', but is certainly not a point.