MAT 102 SPRING 2008
HINTS ON HOMEWORK 8

§8.3. Part 1

We understand that you may not have done algebra for a while, so you may face some real difficulty when you are integrating rational functions. Below are a few examples of how you could deal with such integrals; in particular, focus on why and how the rational functions are decomposed into partial fractions. They are almost the same as the assigned homework problems, except that some numbers are different. Hopefully that will give you a hint on how to solve the homework problems.

(1)
$$\int \frac{dx}{x^2+3x}$$
First one should see that such a fraction is a rational function, meaning that both the numerator (1 in this case) and the denominator (x^2+3x in this case) are polynomials. This suggests that techniques in handling rational functions should be used. There are two main techniques: the first one being an reduction (say by completing the square) to one of those formula in Section 8.1 (or Handout 3), which is applicable whenever the denominator cannot be factorized, and the second one being decomposition into partial fractions, which is applicable whenever the denominator can be factorized. So let's do this integral.

First, the denominator of the integral is x^2+3x. Can it be factorized? In other words, can you write it as product of other (lower-degree) polynomials? (If you are unfamiliar with factorization, stop here and browse the corresponding webpage that I have posted about factorization of polynomials to remind yourself of how that works; we will just need some simple factorizations for the moment.)

Yes, we have $x^2+3x = x(x+3)$. So we would like to use the partial fraction technique. In other words, we want to write $\frac{1}{x^2+3x}$ as combinations of $\frac{1}{x}$ and $\frac{1}{x+3}$. So what do we do?

Let
$$\frac{1}{x^2+3x} = \frac{A}{x} + \frac{B}{x+3}$$
where A and B are constants (meaning that they are numbers that does not depend on x). We want to figure out what A and B are. Let's simplify the right hand side and compare with the left-hand side. The right hand side is
$$\frac{A}{x} + \frac{B}{x+3} = \frac{A(x+3)}{x(x+3)} + \frac{Bx}{x(x+3)} = \frac{(A+B)x+3A}{x^2+3x}.$$
So to make it match with the left hand side, we need
$$\begin{cases} A+B=0 \\ 3A=1. \end{cases}$$

Solving, we get

$$A = \frac{1}{3} \quad \text{and} \quad B = -\frac{1}{3}$$

so

$$\frac{1}{x^2+3x} = \frac{1}{3x} - \frac{1}{3(x+3)}.$$

(This can also be checked directly if you just compute what the right hand side is.) From here we can calculate the given integral, because the right hand side is easy to integrate:

$$\int \frac{dx}{x^2+3x} = \frac{1}{3}\int \frac{dx}{x} - \frac{1}{3}\int \frac{dx}{x+3} = \frac{1}{3}(\ln|x| - \ln|x+3|) + C.$$

This completes the solution.

(2) Try now

$$\int \frac{dx}{x^2+2x}.$$

(3)

$$\int \frac{(x+4)dx}{x^2-x}$$

Again, this is a rational function. Let's check whether the denominator can be factorized:

Yes! $x^2 - x = x(x-1)$. So we want to do partial fraction:

Let

$$\frac{x+4}{x^2-x} = \frac{A}{x} + \frac{B}{x-1}.$$

(Why?) Then the right hand side is

$$\frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)}{x(x-1)} + \frac{Bx}{x(x-1)} = \frac{(A+B)x - A}{x^2-x}$$

so comparing coefficients, we get

$$\begin{cases} A + B = 1 \\ -A = 4. \end{cases}$$

i.e. $A = -4$, $B = 5$. So

$$\frac{x+4}{x^2-x} = -\frac{4}{x} + \frac{5}{x-1}.$$

Integrating, we get

$$\int \frac{(x+4)dx}{x^2-x} = -4\int \frac{dx}{x} + 5\int \frac{dx}{x-1} = -4\ln|x| + 5\ln|x-1| + C.$$

(4) Try now

$$\int \frac{(y+4)dy}{y^2+y}.$$

(5) So the success of the whole method relies on two things: first that you can factorize the denominator, and second that you can decompose the given rational function into (simpler) partial fractions that you can handle. The first bit could be hard if you haven't done it for a while. Here is a brief review of what you should know:

- Factorizing out any common factors in x (e.g. $x^2 + 3x$, $x^3 + x$)
- Factorizing difference of squares (e.g. $x^2 - 4$, $x^2 - 1$, $4x^2 - 9$)
- Factorizing a general quadratic polynomial (e.g. $x^2 - 7x + 12$, $4x^2 - 4x + 1$)

If you have no trouble factorizing any of these you should basically be fine. If you have difficulty with any of them, read the webpage about factorization before you proceed.

(6) Now a slightly more difficult one:

$$\int \frac{(3x+10)dx}{x^2+7x+12}$$

First, factorize the denominator: $x^2+7x+12=(x+3)(x+4)$.

Next, decompose the given fraction into partial fractions: Let

$$\frac{3x+10}{x^2+7x+12}=\frac{A}{x+3}+\frac{B}{x+4}.$$

The right hand side is

$$\begin{aligned}\frac{A}{x+3}+\frac{B}{x+4}&=\frac{A(x+4)}{(x+3)(x+4)}+\frac{B(x+3)}{(x+3)(x+4)}\\&=\frac{(A+B)x+(4A+3B)}{x^2+7x+12}.\end{aligned}$$

Comparing coefficients, we get

$$\begin{cases}A+B=3\\4A+3B=10.\end{cases}$$

It's not hard to solve this pair of equations: in case you have forgotten, here is how to do it. By the first equation, $A=3-B$, so we can substitute this into the second one, and get

$$4(3-B)+3B=10$$

(we can do this substitution because both equations are simultaneously true). In any event, we get $12-B=10$, i.e. $B=2$, so $A=3-B=3-2=1$, or in other words the solution is $A=1$ and $B=2$. Hence now

$$\frac{3x+10}{x^2+7x+12}=\frac{1}{x+3}+\frac{2}{x+4}.$$

Integrating, we get

$$\int \frac{(3x+10)dx}{x^2+7x+12}=\int\frac{dx}{x+3}+2\int\frac{dx}{x+4}=\ln|x+3|+2\ln|x+4|+C.$$

(7) Try now

$$\int \frac{(2x+1)dx}{x^2-7x+12}.$$

(8) If you have taken the pain to go through all the above and reached here, you probably deserve a break. Here is a joke:

Q: Why do you rarely find mathematicians spending time at the beach?

A: Because they have sine and cosine to get a tan and don't need the sun!

Read on...

(9) Sometimes you have to factorize a little bit more:

$$\frac{3x+1}{x^3-x}.$$

Let's factorize the denominator: $x^3-x=x(x^2-1)$. But wait, this is not the end yet! Because x^2-1 can be further factorized: $x^2-1=(x-1)(x+1)$. So altogether,

$$x^3-x=x(x-1)(x+1).$$

The game is that you will always have to factorize as completely as possible before you do partial fractions.

Now we do partial fractions: Let

$$\frac{3x+1}{x^3-x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

Then the right hand side is equal to

$$\begin{aligned}
&\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \\
=&\frac{A(x-1)(x+1)}{x(x-1)(x+1)} + \frac{Bx(x+1)}{x(x-1)(x+1)} + \frac{Cx(x-1)}{x(x-1)(x+1)} \\
=&\frac{A(x^2-1) + B(x^2+x) + C(x^2-x)}{x^3-x} \\
=&\frac{(A+B+C)x^2 + (B-C)x - A}{x^3-x}.
\end{aligned}$$

So we get

$$\begin{cases} A+B+C=0 \\ B-C=3 \\ -A=1. \end{cases}$$

Hence $A=-1$, and

$$\begin{cases} -1+B+C=0 \\ B-C=3. \end{cases}$$

Solving, we get $B=2$ and $C=-1$. So

$$\frac{3x+1}{x^3-x} = -\frac{1}{x} + \frac{2}{x-1} - \frac{1}{x+1}.$$

Integrating, we get

$$\begin{aligned}
\int \frac{(3x+1)dx}{x^3-x} &= -\int \frac{dx}{x} + 2\int \frac{dx}{x-1} - \int \frac{dx}{x+1} \\
&= -\ln|x| + 2\ln|x-1| - \ln|x+1| + C.
\end{aligned}$$

(10) Try now

$$\int \frac{(x+3)dx}{2x^3-8x}.$$

§8.3. Part 2

Now that you have learned the basics of partial fractions, there are 3 more tricks that you need to know. (Just three, I promise.) They are:

- what happens if you have a repeated linear factor in the denominator
- what happens if you have a quadratic factor in the denominator that you cannot factorize
- what happens if you have an improper fraction (namely that the degree of the polynomial in the numerator is bigger than or equal to that of the denominator)

Let's deal with them one by one.

(1)

$$\int \frac{xdx}{x^2+2x+1}$$

Factorize the denominator: we get $x^2+2x+1=(x+1)^2$. Therefore the fraction that we are dealing with is

$$\frac{x}{(x+1)^2}.$$

This is tricky because the factor $x+1$ appears twice in the denominator: whenever we do partial fraction for these quotients, we need to (and this is the only little extra bit that we need to know) decompose the fraction not only in terms of $\frac{1}{x+1}$, but also in terms of $\frac{1}{(x+1)^2}$. In other words, what we now do is the following:

Let

$$\frac{x}{x^2+2x+1}=\frac{A}{x+1}+\frac{B}{(x+1)^2}.$$

Then proceed just as usual: the right hand side is

$$\frac{A}{x+1}+\frac{B}{(x+1)^2}=\frac{A(x+1)}{(x+1)^2}+\frac{B}{(x+1)^2}=\frac{Ax+(A+B)}{x^2+2x+1}.$$

Hence we get

$$\begin{cases} A=1 \\ A+B=0. \end{cases}$$

i.e. $A=1$ and $B=-1$. Hence

$$\frac{x}{x^2+2x+1}=\frac{1}{x+1}-\frac{1}{(x+1)^2}.$$

Integrating, we get

$$\int\frac{xdx}{x^2+2x+1}=\int\frac{dx}{x+1}-\int\frac{dx}{(x+1)^2}=\ln|x+1|+\frac{1}{x+1}+C.$$

Remember how to compute $\int\frac{dx}{(x+1)^2}$? (Hint: use substitution.)

The only trick here now is that you have to get the form of the partial fraction correct. You couldn't solve the problem at all if you didn't know the partial fraction of $\frac{x}{x^2+2x+1}$ is going to look like

$$\frac{A}{x+1}+\frac{B}{(x+1)^2}.$$

The rule is the following: whenever $(2x+3)^2$ appears in the denominator, use both $\frac{1}{2x+3}$ and $\frac{1}{(2x+3)^2}$; in other words, your partial fraction will involve terms like

$$\frac{A}{2x+3}+\frac{B}{(2x+3)^2}.$$

Whenever $(2x+3)^3$ appears in the denominator, you use all of

$$\frac{1}{2x+3},\quad \frac{1}{(2x+3)^2}\quad \text{and}\quad \frac{1}{(2x+3)^3};$$

in other words, your partial fraction will look like

$$\frac{A}{2x+3}+\frac{B}{(2x+3)^2}+\frac{C}{(2x+3)^3}.$$

Of course there is nothing special about 2 and 3 in $2x+3$; any other numbers would work.

(2) Try now

$$\int\frac{xdx}{x^2-4x+4}.$$

(3)
$$\int \frac{4x^2dx}{(x+1)(x^2-2x+1)}$$
The denominator can be factorized: it was already partially done for you. All that you need to do is to factorize it completely: so you want to factorize x^2-2x+1 into $(x-1)^2$ and the whole denominator is just $(x+1)(x-1)^2$. Now do partial fraction: you get terms like
$$\frac{1}{x+1}, \quad \frac{1}{x-1}, \quad \text{and} \quad \frac{1}{(x-1)^2}.$$
(Why?)
Now let
$$\frac{4x^2}{(x+1)(x^2-2x+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$
Then the right hand side is equal to
$$\begin{aligned}&\frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\\ =&\frac{A(x-1)^2}{(x+1)(x-1)^2} + \frac{B(x+1)(x-1)}{(x+1)(x-1)^2} + \frac{C(x+1)}{(x+1)(x-1)^2}\\ =&\frac{A(x^2-2x+1)+B(x^2-1)+C(x+1)}{(x+1)(x-1)^2}\\ =&\frac{(A+B)x^2+(-2A+C)x+(A-B+C)}{(x+1)(x^2-2x+1)}.\end{aligned}$$
So we get
$$\begin{cases} A+B=4\\ -2A+C=0\\ A-B+C=0.\end{cases}$$
Hence from the second equation $C=2A$, and from the third we have $B=A+C=A+2A=3A$. Plugging into the first equation, $A+3A=4$, so $A=1$, and then $B=3A=3$, $C=2A=2$. It follows that
$$\frac{4x^2}{(x+1)(x^2-2x+1)} = \frac{1}{x+1} + \frac{3}{x-1} + \frac{2}{(x-1)^2}.$$
Integrating, we get
$$\begin{aligned}\int \frac{4x^2dx}{(x+1)(x^2-2x+1)} &= \int \frac{dx}{x+1} + \int \frac{3dx}{x-1} + \int \frac{2dx}{(x-1)^2}\\ &= \ln|x+1| + 3\ln|x-1| - \frac{2}{x-1} + C.\end{aligned}$$

(4) Try now
$$\int \frac{x^2dx}{(x-1)(x^2+2x+1)}.$$

(5) Joke time!
Q: How do you make one burn?
A: Differentiate a log fire!

(6) Next, if there is something that you cannot factorize in the denominator: actually you can always factorize things (at least in principle) until you get quadratic factors, and whenever you cannot factorize a quadratic polynomial that appears in the denominator, one of those little formula in Section

8.1 (or Handout 3, Section 4) would help. This is indeed the easy case: you can just plug formula and you are done.

(7)
$$\int \frac{dx}{x^2 + 2x + 2}.$$

You have seen this already. You cannot factorize $x^2 + 2x + 2$ because the discriminant is negative (if you know what that is). In any case you will know that this is something that you cannot factorize, and you try completing the square: you get
$$\int \frac{dx}{(x+1)^2 + 1}.$$
So if you now make the change of variable $u = x + 1$, you end up with the integral
$$\int \frac{du}{u^2 + 1}$$
which you can evaluate by a formula. So the answer is just
$$\begin{aligned}\int \frac{dx}{x^2 + 2x + 2} &= \int \frac{dx}{(x+1)^2 + 1} \\ &= \int \frac{du}{u^2 + 1} \\ &= \tan^{-1} u + C \\ &= \tan^{-1}(x+1) + C.\end{aligned}$$

(8) Try now
$$\int \frac{dx}{x^2 - 4x + 5}.$$

(9) However, sometimes in the denominator you run into a product of quadratic expressions that you cannot factorize, with some other say linear factors. e.g.
$$\int \frac{3t^2 + 3t + 1}{t^3 + t} dt.$$
Here things gets a little complicated, but not too much: you just have to realize two things. First, you still want to factorize the denominator as much as possible. So you get
$$t^3 + t = t(t^2 + 1).$$
If $t^2 + 1$ could be further factorized, you should indeed factorize it and use partial fraction, but since it actually couldn't be factorized, the best you can do is to do partial fraction from here. So the second thing that you need to know is that the partial fraction that you will get will be of the form
$$\frac{A}{t} + \frac{Bt + C}{t^2 + 1}.$$
In other words, on the top of $t^2 + 1$, you get not only a number, but a linear factor in t, namely $Bt + C$. As long as you pick a partial fraction of this form you will be able to compute the partial fraction decomposition:

Let
$$\frac{3t^2 + 3t + 1}{t^3 + t} = \frac{A}{t} + \frac{Bt + C}{t^2 + 1}.$$

Then the right hand side is equal to

$$\frac{A}{t}+\frac{Bt+C}{t^2+1}=\frac{A(t^2+1)}{t(t^2+1)}+\frac{(Bt+C)t}{t(t^2+1)}=\frac{(A+B)t^2+Ct+A}{t^3+t}$$

so comparing coefficients, we get

$$\begin{cases} A+B=3 \\ C=3 \\ A=1. \end{cases}$$

This says $A=1$, $B=2$ and $C=3$, so

$$\frac{3t^2+3t+1}{t^3+t}=\frac{1}{t}+\frac{2t+3}{t^2+1}.$$

You want to integrate the right hand side here, and the first term is alright; you know how to integrate $\frac{1}{t}$. The trouble comes with the second term. However, this is not very hard: recall from Section 8.1 (or Handout 3, Section 4) that there are some basic integrals that you can evaluate. One of them is

$$\frac{1}{t^2+1}.$$

There is a formula for that. Another one is

$$\frac{t}{t^2+1}.$$

You can do this by substitution. So all in all, to compute the integral

$$\int \frac{2t+3}{t^2+1}dt$$

it suffices to split the integral into the sum of two terms: just do

$$\int \frac{2t+3}{t^2+1}dt=\int \frac{2tdt}{t^2+1}+\int \frac{3dt}{t^2+1}$$

and evaluate each of them separately. Together, we get

$$\begin{aligned}\int \frac{3t^2+3t+1}{t^3+t}dt &= \int \frac{1}{t}dt+\int \frac{2tdt}{t^2+1}+\int \frac{3dt}{t^2+1} \\ &= \ln|t|+\ln(t^2+1)+3\tan^{-1}t+C.\end{aligned}$$

(10) Try now

$$\int \frac{3t^2+t+4}{t^3+t}dt.$$

(11) Finally, sometimes you run into a rational function in which the degree of the polynomial on the numerator is bigger than or equal to the degree of the polynomial on the denominator. In this case, you just need to do one extra step: Always perform a long division first to simplify the fraction into a sum of a polynomial with another rational function whose numerator has degree smaller than the degree of the denominator. This is really like if you have a fraction of numbers $\frac{11}{2}$ you really want to write this as $5+\frac{1}{2}$ - you want to split out the 'integral part' of the fraction because it makes the fraction more transparent and easier to deal with. Let's go through the following (last) example together:

$$\int \frac{x^3dx}{x^2+2x+1}.$$

First, the power of the numerator is bigger than that of the denominator. So we want to do a long division: we want to divide x^3 by x^2+2x+1. Step back and take a look at the webpage I posted online if you are not sure how to do this. This is really easier than you might have thought; it is almost like how you carry out a long division of numbers. The answer in this case is the following: the quotient is $x-2$, the remainder is $3x+2$. Hence

$$\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{x^2+2x+1}.$$

(Think about $\frac{11}{2} = 5 + \frac{1}{2}$: it is just another way of saying that when you divide 11 by 2, you get a quotient of 5 and a remainder of 1.)

Now we want to integrate this fraction. The first term $x-2$, which does not involve a quotient, is of course easy to integrate; we just get

$$\int (x-2)dx = \frac{1}{2}x^2 - 2x + C.$$

Do you see now why we want to split this 'integral bit' out now?

The second term is just slightly harder: you just want to repeat what you have done so far when you integrate the other proper fractions above. Here is how you do it:

First step: factorize the denominator as far as possible. Here this means you write $x^2+2x+1 = (x+1)^2$.

Second step: decompose

$$\frac{3x+2}{x^2+2x+1}$$

into a partial fraction. Now we have a repeated linear factor in the denominator! So we would find the partial fraction decomposition by letting

$$\frac{3x+2}{x^2+2x+1} = \frac{A}{x+1} + \frac{B}{(x+1)^2}.$$

Then the right hand side is

$$\frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1)}{(x+1)^2} + \frac{B}{(x+1)^2} = \frac{Ax+(A+B)}{(x+1)^2}.$$

Comparing coefficients, we get

$$\begin{cases} A = 3 \\ A + B = 2 \end{cases}$$

so $A = 3$, $B = -1$. In other words,

$$\frac{3x+2}{x^2+2x+1} = \frac{3}{x+1} - \frac{1}{(x+1)^2}.$$

Integrating, we get

$$\begin{aligned} \int \frac{3x+2}{x^2+2x+1}dx &= \int \frac{3}{x+1}dx - \int \frac{1}{(x+1)^2}dx \\ &= 3\ln|x+1| + \frac{1}{x+1} + C. \end{aligned}$$

Together with the easy term, we get

$$\begin{aligned} \int \frac{x^3dx}{x^2+2x+1} &= \int (x-2)dx + \int \frac{3x+2}{x^2+2x+1}dx \\ &= \frac{1}{2}x^2 - 2x + 3\ln|x+1| + \frac{1}{x+1} + C. \end{aligned}$$

Just a final quick tips: your remainder will always have degree strictly smaller than your denominator.

(12) Try now

$$\int \frac{x^3dx}{x^2+2x+1}.$$

Hope this helps you a little bit when you work on your homework this week!

P.S. Can you summarize, in your own words, what you have learned from the above examples?