MAT 102 SPRING 2008
HANDOUT 3 - A COMPREHENSIVE REVIEW OF INTEGRALS

When you revise what we have done so far, it helps a lot if you could organize things into little packages, each of which you can handle. Here is an attempt to help you organize the integrals that you know: there are of course lots of variants that are not listed here, but basically the most common integrals are all implicitly in the list below.

(Definite integrals are not listed, but it should now be an easy business to compute a definite integral if you know how to compute the indefinite ones.)

1. The basics

These are those whose anti-derivatives can be evaluated by observation:

- $\int x^m dx$ (including $\int \frac{dx}{x}$, $\int \sqrt{x}dx$, $\int \frac{dx}{\sqrt{x}}$, $\int (2x-7)^m dx$, etc)
- $\int e^x dx$ (including $\int e^{-2x} dx$ etc)
- $\int \sin x dx$, $\int \cos x dx$ (including $\int \sin(3-7x)dx$, etc)
- $\int \sec^2 x dx$, $\int \csc^2 x dx$ and their variants
- $\int \sec x \tan x dx$, $\int \csc x \cot x dx$ and their variants

From now on, the variants of the integrals will be understood to be computable and I will leave those to you.

2. Basic Substitutions

Here come the basic substitutions. It is hard now to list them all, but there are certain patterns that you can observe, and let's put them in the following form, where f is just some nice function that you can integrate:

- $\int f(\sin x) \cos x dx$ (e.g. $\int e^{\sin x} \cos x dx$, $\int (\sin^3 x + \sin^2 x + 1) \cos x dx$ or $\int \frac{\cos x dx}{\sin^2 x}$)
- $\int f(\cos x) \sin x dx$ (I will leave it to you to construct some examples that are of this form!)
- $\int f(\tan x) \sec^2 x dx$, $\int f(\cot x) \csc^2 x dx$
- $\int f(x^2) x dx$ (e.g. $\int e^{x^2} x dx$, $\int \sin(2x^2 - 7) x dx$, $\int \frac{x dx}{\sqrt{1+x^2}}$)
- More generally, $\int f(x^m) x^{m-1} dx$ (Any examples?)

In particular, you can now compute

- $\int \tan x dx$, $\int \cot x dx$

3. Trigonometric identities

If you see a trigonometric function, you may want to use some of the trigonometric identities. Here they are again:

$$\tan x = \frac{\sin x}{\cos x}$$
$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x = \sec^2 x - 1$$
$$\cot^2 x = \csc^2 x - 1$$
$$\sin 2x = 2 \sin x \cos x$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

In particular, you will be able to integrate:

- $\int \tan^2 x dx$, $\int \cot^2 x dx$
- $\int \sin^2 x dx$, $\int \cos^2 x dx$
- $\int \sqrt{1 + \cos 2x} dx$, $\int \sqrt{1 - \cos 2x} dx$

You will also be able to integrate combinations of these, like $\int (\sec x + \tan x)^2 dx$.

There are two more that stands out all the time, and requires a rather tricky formulat to integrate:

- $\int \sec x dx$, $\int \csc x dx$

That is going to take care of all the trigonometric integrals that we encounter so far.

4. Quotients of polynomials (or their square roots)

Here come some integrals that are implicit in the book, but often handy to know. I am just going to put down the form of the integrals and organize them into patches so that you can get a hold of them. The variations will be mostly left to you.

First, the ones whose anti-derivatives are known:

- $\int \frac{dx}{\sqrt{1-x^2}}$
- $\int \frac{dx}{1+x^2}$
- $\int \frac{dx}{\sqrt{1+x^2}}$
- $\int \frac{dx}{\sqrt{x^2-1}}$
- $\int \frac{dx}{x\sqrt{x^2-1}}$

So for the first time, it is important whether it is $1 - x^2$ or $x^2 - 1$ or $x^2 + 1$!

Incidentally, all these can be evaluated with a correct trigonometric substitution ($x = \sin t$ for the first one, $x = \tan t$ for the second, $x = \sinh t$ for the third, $x = \cosh t$ for the fourth and $x = \sec t$ for the last). That helps you reduce the number of formula you have to remember.

As a matter of fact, these are not all, but chances are that you will rarely meet the other ones, at least for now.

Second, let's say the denominator is a quadratic polynomial. There are several possibilities for the numerator.

- $\int \frac{dx}{1+x^2}$, $\int \frac{xdx}{1+x^2}$, $\int \frac{x^2dx}{1+x^2}$

Each of them needs to be dealt with separately, with a different method. Do you see how? (Formula for the first one, substitution for the second - we have seen this already, and finally reducing the improper fraction for the third.)

Their variants, like $\int \frac{dx}{1+(2x-3)^2}$, can be handled as long as you can handle the original three.

It may happen that the denominators are $1-x^2$ (or x^2-1), but those are harder and we will take care of them later.

Third, let's say the denominator is the square root of a quadratic polynomial:

- $\int \frac{dx}{\sqrt{1-x^2}}$, $\int \frac{xdx}{\sqrt{1-x^2}}$, $\int \frac{x^2dx}{\sqrt{1-x^2}}$
- $\int \frac{dx}{\sqrt{1+x^2}}$, $\int \frac{xdx}{\sqrt{1+x^2}}$, $\int \frac{x^2dx}{\sqrt{1+x^2}}$
- $\int \frac{dx}{\sqrt{x^2-1}}$, $\int \frac{xdx}{\sqrt{x^2-1}}$, $\int \frac{x^2dx}{\sqrt{x^2-1}}$

These are the most common ones. For the first one in each row, use formula. For the second one in each row, use a substitution. For the third one in each row, use a reduction of improper fraction.

Of course their variants can now be integrated as well.

Finally, a few extra tricks in dealing with the current class of integrals:

4.1. **Splitting into a sum.** The above integrals can of course easily combine to give expressions that look impossible to integrate at first sight. But don't be scared; these are often the easiest ones that you get!

- e.g. $\int \frac{(2x-9)dx}{\sqrt{1+x^2}}$ is just the sum of two terms, each of which can be dealt with according to what we have above.

4.2. **Completing the square.** If you have a quadratic polynomial (or its square root) in the denominator and that quadratic polynomial has a linear term (like $2x$), you will want to use completing the square to reduce it to one of the above forms.

- e.g. $\int \frac{dx}{x^2+2x+5}$, $\int \frac{dx}{\sqrt{4x^2-4x+5}}$

4.3. **More advanced substitution.** Sometimes you have to make a substitution before you are reduced to the integrals above.

- e.g. $\int \frac{\cos x dx}{\sqrt{1+\sin^2 x}}$, $\int \frac{e^x dx}{\sqrt{1-e^{2x}}}$.

These are harder, but they could be carried out with patience.

5. Integration by parts

Finally, as of now, the last thing you know about integrals is that you can do integration by parts if you see the product of two functions, one of which has an antiderivative that you can compute.

- e.g. $\int x \sin x dx$, $\int xe^x dx$, $\int x^2 \cos x dx$, $\int \ln x dx$, $\int e^x \sin x dx$.