MAT 102 SPRING 2008
HANDOUT 4 - COMPREHENSIVE REVIEW OF INTEGRALS II

This is a continuation from Handout 3 and summarizes the integrals that you have learned after learning integration by parts. The integrals correspond to those appearing in Sections 8.3 to 8.5 of the book.

Again, all the following integrals have a lot of simple variants, and by now hopefully it is a simple matter to reduce those to the list below. Also, definite integrals are not listed, but it shouldn't be hard to compute definite integrals if you know the indefinite ones.

1. Rational functions

In Section 8.1 we have already seen how to integrate some functions that are quotients of polynomials (or square roots of polynomials). Recall Section 4 of Handout 3 if necessary. Those represent integrals whose denominators cannot be factorized; if it happens that you can factorize the denominators, the technique of partial fraction should be used.

A few basic rules:

- If the degree of the numerator is bigger than or equal to the degree of the denominator, always carry out a long division first.
- If the denominator can be factorized, use partial fraction - and remember to always factorize *completely.*
- If the denominator has a repeated linear factor (e.g. $(x+1)^2$), the form of the partial fraction has to be suitably modified (e.g. need
$$\frac{A}{x+1} + \frac{B}{(x+1)^2}$$
in the above example).
- If the denominator has a quadratic factor that cannot be factorized (e.g. $x(x^2+1)$), the form of the partial fraction should also be modified accordingly (e.g. need
$$\frac{A}{x} + \frac{Bx+C}{x^2+1}$$
in the above example).

See hints for HW 8 for a detailed discussion.

Some typical examples:

$$\int \frac{(x+4)dx}{x^2-x} \qquad \int \frac{(3x+10)dx}{x^2+7x+12} \qquad \int \frac{3x+1}{x^3-x}dx \qquad \int \frac{xdx}{x^2+2x+1}$$

$$\int \frac{4x^2dx}{(x+1)(x^2-2x+1)} \qquad \int \frac{3t^2+3t+1}{t^3+t}dt \qquad \int \frac{x^3dx}{x^2+2x+1}$$

2. Trigonometric Substitutions

These are substitutions to treat integrals involving $a^2 + x^2$, $a^2 - x^2$ and $x^2 - a^2$ (in particular their square roots). These are based on the following trigonometric identities:

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \sinh^2\theta = \cosh^2\theta$

To apply them, remember the following rules:

- To deal with integrals involving $\sqrt{1-x^2}$, use the substitution $x = \sin\theta$ and hope that it works. This is plausible because then $\sqrt{1-x^2} = \cos\theta$.
- To deal with integrals involving $1+x^2$, use one of the following substitutions: $x = \tan\theta$ or $x = \sinh\theta$, and see which one works. This is plausible because $1 + x^2$ is then $\sec^2\theta$ in the first case, and $\cosh^2\theta$ in the second.
- To deal with integrals involving $\sqrt{x^2-1}$, use one of the following substitutions: $x = \sec\theta$ or $x = \cosh\theta$. This is plausible because $\sqrt{x^2-1}$ is then $\tan\theta$ in the first case, and $\sinh\theta$ in the second.

Try now to evaluate the basic integrals in Section 8.1 (or Section 4 of Handout 3), using these substitutions:

- $\int \frac{dx}{\sqrt{1-x^2}}$
- $\int \frac{dx}{1+x^2}$
- $\int \frac{dx}{\sqrt{1+x^2}}$
- $\int \frac{dx}{\sqrt{x^2-1}}$
- $\int \frac{dx}{x\sqrt{x^2-1}}$

You can now derive these 5 formula even if you forget them!

3. Trigonometric integrals

In Chapter 5 you have already seen some trigonometric integrals. Recall Section 3 of Handout 3 if necessary. The only two main additions here are:

First, you now know how to integrate products of powers of $\sin x$ and $\cos x$:

- $\int \cos^n x \sin^m x dx$ where m, n are non-negative integers.

(Recall how you do it when one of powers is odd, and when both powers are even.)

Actually you also know how to integrate powers of $\tan x$ and $\sec x$.

Next, there are just a few more trigonometric identities that are often useful (adding to the list in Handout 3): they are often called product to sum formula, and they are good because integrating a sum is often a lot easier than integrating a product.

$$\begin{aligned}
\sin A \cos B &= \frac{1}{2}(\sin(A+B) + \sin(A-B)) \\
\cos A \cos B &= \frac{1}{2}(\cos(A+B) + \cos(A-B)) \\
\sin A \sin B &= \frac{1}{2}(\cos(A-B) - \cos(A+B))
\end{aligned}$$

In particular, you will be able to integrate:

- $\int \sin mx \cos nx dx$, $\int \cos mx \cos nx dx$, $\int \sin mx \sin nx dx$, etc

4. Assorted integrals

The final big challenge: You will have to figure out how to evaluate the integrals by yourself - in other words, it's rather unlikely that we tell you explicitly in an exam what method you should use. After all, in real life, when you run into an integral (however likely :p), you will be the one to come up with the method to evaluate the integrals. It's not that hard really; every method has its own merit, and they are applied for good reasons. Just mentally note down why a certain method is good for a particular kind of integrals, and it gets easier with just a little bit of practice.