MAT 218 FALL 2008
FEEDBACK ON PROBLEM SET 10

This problem set seems to be more difficult than the previous ones. Below is a discussion of the common misconceptions, and full solutions to the problems from Spivak.

1. Common errors.

Exercises from Folland.

5.7.2: Make sure you point out what the region S is when you use Stokes' theorem. Usually there are many possible choices of S, and some are more convenient than others. The art here is in making the correct choices.

Also, some of you projected S down and calculated $x^2+y^2+(a-y)^2 = a^2$ in the description of S. Note that the equation just describes the projection of S onto the x, y plane, not the region S itself. The region S should be a disk in the plane $y + z = a$, of radius $a/\sqrt{2}$.

Exercises from Spivak.

4-29: Given a 1-form ω on $[0, 1]$, the question asked you to show that there exists a unique number λ such that there exists a function g for which

$$\omega = \lambda dx + dg$$

and

$$g(0) = g(1).$$

In other words, after choosing λ, you still need to construct g such that the two equations above hold. Almost all of you didn't show the existence of g, nor explain why $g(0) = g(1)$. See solution below.

Also, as some of you correctly pointed out, the assumption $f(0) = f(1)$ in the question is irrelevant.

4-30: Again, one needs to construct also the function g here, and this is a bit more difficult than the corresponding construction in 4.29. See solution below. (I think the hint in the book here is a bit misleading, and there is an easier way of solving the problem without using the hint. I shall, however, also point out below how the hint could be used to solve the problem, as some of you may be interested.)

4-34: Some of you didn't use the definition of the boundary operator ∂ in solving part (a). I know the definition looks complicated to use, but it is indeed the easiest and cleanest way of solving the problem. (Spivak actually made a point about 'good definitions' being easily applicable towards the end of Chapter 4, and this is one instance of that.)

Further problems.

1: Be careful about the signs of the integrals. This involves choosing the correct orientations for the different faces. See the solution to the further problems in problem set 9.

2: Remember when you integrate ϕ on ∂A_1, you actually need parametrize ∂A_1 and integrate the pull-back of ϕ via this parametrization on the space of parameters. In other words, say $c(t) = (1, t, 0)$ is a parametrization of part of ∂A_1, when you integrate ϕ over this part of ∂A_1 you actually need to do

$$\int_0^1 c^*(\phi) = \int_0^1 t dt$$

rather than $\int_0^1 1 dx_1 + x_2 dx_2$ as some of you incorrectly stated.

2. Solution to selected exercises.

Exercises from Spivak.

4-29: Let ω be a 1-form on $[0, 1]$.
Existence. Let $\lambda = \int_0^1 \omega$. Let

$$g(x) = \int_0^x (\omega - \lambda dt).$$

Then

$$dg = \frac{\partial g}{\partial x} dx = \omega - \lambda dx$$

by fundamental theorem of calculus. Also,

$$g(1) = \int_0^1 \omega - \lambda = 0 = g(0)$$

by our choice of λ. This completes the proof of existence of λ and g.
Uniqueness. Suppose λ is a real number such that

$$\omega = \lambda dx + dg$$

for some function with $g(0) = g(1)$. Then integrating both sides from 0 to 1, we get

$$\int_0^1 \omega = \lambda + \int_0^1 dg = \lambda + g(1) - g(0) = \lambda.$$

This determines λ uniquely.
Remark. Actually g is also unique up to an additive constant. Why? (Hint: is there a motivation why we constructed g as we did above?)

4-30: Let ω be a 1-form on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ with $d\omega = 0$.
Method 1.

Let γ be any curve homologous to the unit circle in $\mathbb{R}^2 \setminus \{0\}$. Then

$$\frac{1}{2\pi} \int_\gamma \omega$$

is independent of the choice of γ, because if γ_1 and γ_2 are two such curves, then by Stokes' theorem

$$\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{\partial S} \omega = \int_S d\omega = 0$$

where S is a region in $\mathbb{R}^2 \setminus \{0\}$ whose boundary is $\gamma_1 - \gamma_2$. Let λ be this number. It follows that

$$\int_\eta (\omega - \lambda d\theta) = 0$$

for all piecewise smooth closed curves η. (Why?) Hence if p is a point in $\mathbb{R}^2 \setminus \{0\}$, which we fix from now on, the path integral

$$\int_p^q (\omega - \lambda d\theta)$$

is independent of the path that joins p to q in $\mathbb{R}^2 \setminus \{0\}$. Let $g(q)$ be the above path integral. It then follows that

$$dg = \omega - \lambda d\theta.$$

Remark. Be careful and do NOT conclude that if γ is the unit circle and D is the unit disk then $\int_\gamma \omega = \int_D d\omega = 0$. Why is this not true under our current hypothesis? Can you give an example when this is not true?

Method 2.

Here is a solution that makes use of the hint in the book. Let $c_{R,1}$ be the map $c_{R,1}(t) = (R\cos 2\pi t, R\sin 2\pi t)$ where t runs from 0 to 1. Then by Problem 4-29 above, for each $R > 0$ there exists a number λ_R and a function $g_R(t)$ of one variable such that the pullback of ω under this map satisfies

$$c_{R,1}^*\omega = \lambda_R dt + d(g_R),$$

with $g_R(0) = g_R(1)$. Note that here $d(g_R)$ means $\frac{dg_R}{dt}dt$.

Now note that λ_R is independent of $R > 0$. This is because

$$\begin{aligned}
\lambda_{R_1} - \lambda_{R_2} &= \int_0^1 \lambda_{R_1} dt - \int_0^1 \lambda_{R_2} dt \\
&= \int_{\gamma_{R_1}} \omega - \int_{\gamma_{R_2}} \omega \\
&= \int_{A_{R_1,R_2}} d\omega = 0
\end{aligned}$$

where γ_R is the circle of radius R centered at the origin and A_{R_1,R_2} is the annulus whose boundary is $\gamma_{R_1} - \gamma_{R_2}$. Call this common value $2\pi\lambda$. Then for each $R > 0$ there exists a function $g_R(t)$ of one variable such that

$$c_{R,1}^*\omega = 2\pi\lambda dt + \frac{dg_R}{dt}dt. \qquad (1)$$

Let now (R, θ) be the polar coordinates on $\mathbb{R}^2 \setminus \{0\}$ where θ runs from 0 to 2π. Define a function g on $\mathbb{R}^2 \setminus \{0\}$ by

$$g(R, \theta) = g_R\left(\frac{\theta}{2\pi}\right)$$

using this polar coordinates. Equation (1) now says

$$c_{R,1}^*\left(\omega - \lambda d\theta - \frac{\partial g}{\partial \theta}d\theta\right) = 0.$$

This doesn't say that $\omega - \lambda d\theta - \frac{\partial g}{\partial \theta}d\theta$ is 0; it says that it is a multiple of dr at each point in the punctured plane. Hence there exists a function $h(r, \theta)$ on $\mathbb{R}^2 \setminus \{0\}$ such that

$$\omega = \lambda d\theta + \frac{\partial g}{\partial \theta}d\theta + h(r, \theta)dr.$$

Then since $d\omega = 0$, we have

$$\frac{\partial h}{\partial \theta} = \frac{\partial^2 g}{\partial r \partial \theta}.$$

Hence

$$h(r,\theta) = \frac{\partial g}{\partial r} + k(r)$$

for some function k of r only. Replacing $g(r,\theta)$ by $g(r,\theta) + \int_1^r k(r)dr$, we have

$$h(r,\theta) = \frac{\partial g}{\partial r}$$

without modifying any other properties of g we alluded to above, and it follows that

$$\omega = \lambda d\theta + \frac{\partial g}{\partial \theta} d\theta + \frac{\partial g}{\partial r} dr = \lambda d\theta + dg.$$

4.34(a):

$$\begin{aligned}
\partial C_{F,G} &= \sum_{i=1}^{3} \sum_{\alpha=0,1} (-1)^{i+\alpha} C_{F,G} \circ I^1_{(i,\alpha)} \\
&= -C_{F,G}(0,x,y) + C_{F,G}(1,x,y) \\
&\quad + C_{F,G}(x,0,y) - C_{F,G}(x,1,y) - C_{F,G}(x,y,0) + C_{F,G}(x,y,1).
\end{aligned}$$

Since F_s and G_s are closed curves for all s, the last four terms in the above equation vanishes. It follows that

$$\partial C_{F,G} = -c_{F_0,G_0} + c_{F_1,G_1}.$$