MAT 218 FALL 2008
FEEDBACK ON PROBLEM SET 1

1. Some common mistakes.

Part 2 Q1: Many of you said something like 'since the set $\{0\}$ is closed, it is not open'. But beware: sometimes a closed set could actually be open: for instance, the empty set and the whole $\mathbb{R}$.

§1.4, Q6: Some of you thought (and indeed tried to prove) that a is an accumulation point of S if and only if there is a sequence $\{x_k\}$ in S that converges to a. But this is not true. For example, if $S = \{0\}$, then there is certainly a sequence in S that converges to 0 (namely the constant sequence $x_k = 0$ for all k), but it is not an accumulation point, because any neighborhood of 0 consists of just one single point (and not infinitely many) in S, namely 0. The confusion might have arisen because some of you are so used to thinking that different terms in the sequence are automatically not equal to one another. But this is not quite true; there is such a thing as a constant sequence, and it is precisely our enemy in this situation.

§1.4, Q7: Some of you thought that any boundary point of a set S is an accumulation point of that set as well. This is not quite true: for instance, if $S = \{0\}$, then 0 is a boundary point of S, but it is not an accumulation point of S, because there couldn't be a sequence in S, all of which are not 0, which converges to 0.

§1.5, Q10: Many of you didn't take the care to make sure that what you've constructed is actually a *subsequence* of the original sequence; many just picked a *subset* of the original sequence and showed that it converged to the limsup and the liminf. Be careful here: by saying that $\{x_{n_j}\}$ is a subsequence we require $n_1 < n_2 < \dots$. It takes a little pain to do this, but sometimes it is worth the pain because there are too many things that are true for subsequences and not for subsets.

2. Solution to selected exercises.

I'll present the solutions of some of the exercises that some of you had trouble with.

§1.4, Q6: Let a be an accumulation point of S. Then every neighborhood of a contains infinitely many points in S. In particular, a couldn't be the only point in that neighborhood, so in every ball of radius $1/k$, we can pick a point $x_k \in S$ with $x_k \neq a$. Hence we have picked a sequence in S, none of which is equal to a, that converges to a.

Suppose now there exists a sequence $\{x_k\}$ in S, none of which is equal to a, that converges to a. Then for any neighborhood U of a, there exists K such that $x_k \in U$ for all $k \geq K$. The set $\{x_k \colon k \geq K\}$ must contain infinitely many (different) elements of S: otherwise there would exist a finite set F such that $x_k \in F$ for all k, and with none of the x_k equal to a we can assume $a \notin F$. Then x_k couldn't possibly converge to a, contrary to our choice of $\{x_k\}$. Hence we have exhibited infinitely many elements of S inside U. This completes the proof.

§1.4, Q7: Let S' be the set of accumulation points. We shall first show $\overline{S} \subseteq S \cup S'$. If a is a boundary point of S, then there exists a sequence x_k in S that converges to a. If further that a is not in S, then none of these x_k's could be a, so we have exhibited a sequence in S, none of which is equal to a, that converges to a. By problem 1.4.6, $a \in S'$. This proves $\partial S \setminus S \subseteq S'$, so $\overline{S} \subseteq S \cup S'$.

We next show the reverse inclusion. It suffices to show $S' \subseteq \overline{S}$. Suppose $a \in S'$. Then by problem 1.4.6 again, there exists a sequence $\{x_k\}$ in S that converges to a. Hence by theorem 1.14 in Folland, $a \in \overline{S}$, and we are done.

§1.7, Q5: Let S be an open set. Suppose it is disconnected. Then it can be written as

$$S = A \cup B$$

for some non-empty sets A and B, such that

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset.$$

We claim both A and B are open. Indeed

$$S \setminus \overline{B} = A \setminus \overline{B} = A,$$

so A is the complement of a closed set in an open set, and hence open. Similarly B is open. Hence S is a union of two disjoint non-empty open sets.

Suppose now S is the union of two non-empty disjoint open sets U and V. We claim

$$\overline{U} \cap V = U \cap \overline{V} = \emptyset$$

so that S is disconnected. But the complement of U is a closed set containing V, so $\overline{V}$, being the smallest closed set containing V, must be contained in the complement of U. Hence $U \cap \overline{V} = \emptyset$, and similarly $\overline{U} \cap V = \emptyset$.