MAT 218 FALL 2008
FEEDBACK ON PROBLEM SET 4

1. Some common mistakes.

Folland 2.8.5: Some of you thought that the second directional derivative of a function f in the direction u is given by

$$\sum_{i=1}^{n} u_i^2 f_{ii}$$

but this is not true. (It doesn't agree with the given formula either.)

Folland 2.9.12: It seems that when you use Lagrange's multipliers, most of you don't know how to check whether the point you obtained is a local maximum (or local minimum or neither). Some of you tried to say that if a 'critical point' is not a local minimum then it must be a local maximum, but this is not true. For this one (and the one in Folland 2.9.18) there is an easy way. Please see solution below, and compare with the solution of Folland 2.9.6.

Folland 3.1.5: Some of you thought you need $\nabla G(x, y) \neq 0$ to solve y as a function of x near a point where $G(x, y) = 0$. This is not true. You just need $\partial_y G(x, y) \neq 0$ to apply the implicit function theorem. (One easy way to remember this is to always think about the unit circle in the plane: you can write y as a function of x when (x, y) lies on the unit circle as long as the tangent to the circle at that point is not vertical, i.e. as long as $\partial_y(x^2 + y^2 - 1) = 2y \neq 0$.)

2. Solution to selected exercises.

Folland 2.8.5: Just use

$$\partial_u^2 f = \nabla(\nabla f \cdot u) \cdot u$$

and compute.

Folland 2.9.12: The problem is to minimize

$$f(x, y, z) = x + y + z$$

subject to the condition that

$$g(x, y, z) := xyz - V = 0 \quad \text{and} \quad x, y, z > 0.$$

First notice that if

$$S := \{(x, y, z) \in \mathbb{R}^3 : xyz - V = 0 \text{ and } x, y, z > 0\}$$

then S is a hypersurface (can you draw it?), and we are just minimizing f on S. Next observe that while S is not compact, as the point $(x, y, z) \in S$ goes off to infinity, $f(x, y, z) \to \infty$ as well. (Can you formulate this more precisely?) Hence f cannot have an absolute maximum, and by continuity of f we know an absolute minimum must exist in S. The absolute minimum must be in the interior of S, and thus it must satisfy the Lagrange multiplier equations. Hence at the absolute minimum of f over S, we must have $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$, i.e.

$$(1, 1, 1) = \lambda(yz, xz, xy).$$

It follows that $x = y = z$ at the absolute minimum of f over S, but there is just one point on S that satisfies this, namely $(V^{1/3}, V^{1/3}, V^{1/3})$. Hence the absolute minimum of f over S is given by

$$f(V^{1/3}, V^{1/3}, V^{1/3}) = 3V^{1/3}.$$

Folland 2.9.18: Suppose $c > 0$,

$$f(x_1, x_2, \dots, x_n) = x_1 x_2 \dots x_n,$$

and

$$g(x_1, x_2, \dots, x_n) = x_1 + x_2 + \cdots + x_n.$$

The problem is to maximize $f(x)$ subject to the conditions that

$$g(x) = c \quad \text{and} \quad x_1, x_2, \dots, x_n \geq 0.$$

First notice that if

$$S := \{x \in \mathbb{R}^n : g(x) = c \text{ and } x_1, x_2, \dots, x_n \geq 0\}$$

(note S is a hypersurface because $\nabla g \neq 0$ on S) then we are just maximizing f on the set S, which is compact. Hence by continuity of f we know an absolute maximum must exist in S. The absolute maximum must either be on the boundary or in the interior, and in the latter case it must satisfy the Lagrange multiplier equations. But it is easy to see that the absolute maximum cannot occur on the boundary: simply notice that any point on the boundary of S has one of its coordinates equal to 0, so f is 0 on the boundary of S, while $f > 0$ in the interior of S. Hence at the absolute maximum of f over S, we must have $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$, i.e.

$$\left(\frac{x_1 \dots x_n}{x_1}, \frac{x_1 \dots x_n}{x_2}, \dots, \frac{x_1 \dots x_n}{x_n} \right) = \lambda(1, 1, \dots, 1).$$

It follows that $x_1 = \cdots = x_n$ at the absolute maximum of f over S, but there is just one such point in S, namely $(c/n, \dots, c/n)$. Hence the absolute maximum of f over S must be

$$f\left(\frac{c}{n}, \dots, \frac{c}{n}\right) = \left(\frac{c}{n}\right)^n.$$

Unravelling the notations, this proves the inequality between arithmetic and geometric means:

$$x_1 \dots x_n \leq \left(\frac{x_1 + \cdots + x_n}{n} \right)^n$$

for all non-negative numbers $x_1, \dots, x_n$, with equality if and only if $x_1 = \cdots = x_n$.

(Incidentally, it follows from this argument that the point $(c/n, \dots, c/n)$ is not a local minimum nor a saddle point.)

Spivak 2-41(c): The calculation here is quite involved. Let me put down the essential steps so that those of you who are interested can check for themselves.

Suppose

$$f(x, y) = x(y \log y - y) - y \log x.$$

First, for each fixed $\frac{1}{2} \leq x \leq 2$,

$$\partial_y f(x, y) = x \log y - \log x,$$

so

$$f(x, y) \text{ is increasing in } y \text{ if } y > x^{\frac{1}{x}}$$

and

$$f(x, y) \text{ is decreasing in } y \text{ if } y < x^{\frac{1}{x}}.$$

Now $x^{\frac{1}{x}}$ may or may not lie in $[1/3, 1]$ depending on the value of x. Hence the minimum of $f(x, y)$ over $\frac{1}{3} \leq y \leq 1$ is attained at

$$\begin{aligned} \left(x, \frac{1}{3}\right) &\quad \text{if } \frac{1}{2} \leq x \text{ and } x^{\frac{1}{x}} \leq \frac{1}{3} \\ \left(x, x^{\frac{1}{x}}\right) &\quad \text{if } \frac{1}{3} \leq x^{\frac{1}{x}} \leq 1 \\ (x, 1) &\quad \text{if } 1 \leq x \leq 2. \end{aligned}$$

It follows that

$$\min_{1/3 \leq y \leq 1} f(x, y) = \begin{cases} f\left(x, \frac{1}{3}\right) & \text{if } \frac{1}{2} \leq x \text{ and } x^{\frac{1}{x}} \leq \frac{1}{3} \\ f\left(x, x^{\frac{1}{x}}\right) & \text{if } \frac{1}{3} \leq x^{\frac{1}{x}} \leq 1 \\ f(x, 1) & \text{if } 1 \leq x \leq 2. \end{cases}$$

Then it remains to observe that the above function is decreasing in x when $1/2 \leq x \leq 2$. Hence

$$\max_{1/2 \leq x \leq 2} \left(\min_{1/3 \leq y \leq 1} f(x, y) \right) = f\left(\frac{1}{2}, \frac{1}{3}\right).$$