MAT 218 FALL 2008
FEEDBACK ON PROBLEM SET 6

Most of you did well on this problem set. I will just outline the solutions to some of the harder problems, commenting on the various possible approaches along the way.

1. Solution to selected exercises.

Exercises from Folland.

4.3.7: This is not a really difficult problem, and there are a number of possible solutions. However, I think what is important here is that if you see an integral of the form

$$\int_0^x \int_0^y g(t)dtdy$$

you really need to know how to *simplify* it without first knowing the answer that was given in the problem. One natural way is to use Fubini's theorem:

$$\int_0^x \int_0^y g(t)dtdy = \int_0^y \int_t^x g(t)dydt = \int_0^y (x-t)g(t)dt.$$

Another natural way is to integrate by parts in y:

$$\int_0^x \int_0^y g(t)dtdy = \left(y\int_0^y g(t)dt\right)\Big|_{y=0}^{y=x} - \int_0^x tg(t)dt = \int_0^y (x-t)g(t)dt.$$

Exercises from Spivak.

3-7: There are two ways to approach this problem. The easier way, I think, is the direct approach, that is to construct a partition whose upper sum is as small as one pleases. The other way is to see that the function is continuous almost everywhere; one just needs to show that the function is continuous at every $(x,y) \notin \mathbb{Q}^2$, and that $(\mathbb{Q} \cap [0,1])^2$ has measure zero.

Let me carry out the direct approach here. Given any $\varepsilon > 0$, we shall construct a partition P of $[0,1]^2$ such that the upper sum $U(f,P) < \varepsilon$. Let N be a positive integer such that $1/N < \varepsilon$. Then there are just finitely many rational numbers in $[0,1]$ whose denominator, in reduced form, is smaller than N. Call these rational numbers $y_1, \dots, y_m$. Then $f(x,y) > \varepsilon$ only when $x \in \mathbb{Q}$ and y is one of these y_i's. Let $I_1, \dots, I_m$ be disjoint intervals on the real line such that $y_i \in I_i$ for all i and the total length of these I_i's does not exceed ε. Then $\{[0,1] \times I_i\}_{i=1}^m$ cover the set $\{(x,y) \colon f(x,y) > \varepsilon\}$. We can complete this cover to a partition of $[0,1]^2$ by choosing intervals $J_1, \dots, J_m$ such that $I_1, \dots, I_m, J_1, \dots, J_m$ partition $[0,1]$. Then

$$\{[0,1] \times I_i, [0,1] \times J_i\}_{i=1}^m$$

partition $[0,1]^2$. Call this partition P. Then

$$U(f,P) \le \sum_{i=1}^m |I_i| \sup_{[0,1]\times I_i} f + \sum_{i=1}^m |J_i| \sup_{[0,1]\times I_i} f \le \sum_{i=1}^m |I_i| + \varepsilon = 2\varepsilon.$$

Since $L(f,P)$ is clearly 0, this proves that f is integrable on $[0,1]^2$ and $\int_{[0,1]^2} f = 0$.

3-9: If a set can be covered by finitely many rectangles of finite volumes, then its diameter is finite, so it must be bounded.

Some of you have rather complicated example of a closed set of measure 0 that does not have content 0. But there is a really easy example: just the set of natural numbers (or any unbounded countable set in $\mathbb{R}^n$).

3-18: Again there are two approaches to the problem. Either one follows the hint and shows that $\{x\colon f(x) > 1/n\}$ has measure 0 for each n and takes their union, or one shows that $\{x\colon f(x) \neq 0\}$ is contained in the set of discontinuities of f which has measure 0 because f is (Riemann) integrable. The former proof is slightly better in that it will extend to prove a similar assertion when you study Lebesgue integrals. The latter proof is correct for Riemann integrable functions, but it cannot be extended to Lebesgue integrable ones (although in this course you shouldn't be really worrying about Lebesgue integrals).