MAT 218 FALL 2008
FEEDBACK ON PROBLEM SET 7

Many of you are seeing differential forms for the first time, and it takes a little time to get used to them. However, once you master the formalism, it is a very convenient tool. Therefore I think I will begin with some little tricks in dealing with differential forms - hopefully you'll see how many of the computations you made in the problem set can be simplified.

1. Comments.

Part 2.

1. It is helpful to note that if ω is a 1-form, then

$$\omega \wedge \omega = 0.$$

Therefore if one needs to compute say $d(f(x,y,z)dx \wedge dy)$, one knows right away that it is not necessary to compute the x and y derivatives of f, for $dx \wedge dx = 0$ and $dy \wedge dy = 0$.

Notice, however, that $\omega \wedge \omega$ is NOT zero for 2-forms. Indeed wedge products induce a symmetric bilinear pairing from 2-forms into 4-forms; in other words,

$$\omega_1 \wedge \omega_2 = \omega_2 \wedge \omega_1$$

if ω_1, ω_2 are 2-forms. (But this is not true for 1-forms. Why?) Can you figure out for what values of p is it true that $\omega \wedge \omega = 0$ for all p-forms ω?

Part (c) has been difficult for many of you. See solution below.

3. I know the computations here are rather tedious, but for the computation of the last two differential forms, there is a simple way of doing it: indeed we are just working with spherical coordinates in this exercise, and $t_1^2 = x_1^2 + x_2^2 + x_3^2$ in such coordinates, so

$$\omega_2 = x_1 dx_1 + x_2 dx_2 + x_3 dx_3 = \frac{1}{2} d(x_1^2 + x_2^2 + x_3^2) = \frac{1}{2} d(t_1^2) = t_1 dt_1.$$

Also, $\omega_3 = dx_1 \wedge dx_2 \wedge dx_3$ is just the volume form in the rectilinear coordinates. Therefore in spherical coordinates this is equal to

$$\omega_3 = \det J dt_1 \wedge dt_2 \wedge dt_3 = -t_1^2 \sin t_2 dt_1 \wedge dt_2 \wedge dt_3,$$

where J is the Jacobian matrix of the change of coordinates $(t_1, t_2, t_3) \mapsto (x_1, x_2, x_3)$, which many of you have computed in earlier parts of the problem. If you remember the change of variable formula for spherical coordinates, you also know right away that $\det J$ is $\pm t_1^2 \sin t_2$, where the sign depends on whether the map $(t_1, t_2, t_3) \mapsto (x_1, x_2, x_3)$ is orientation-preserving. In this case the minus sign should be chosen, because the map reverses orientation.

Exercises from Spivak.

4-19. I think this is the most important exercise you have done so far - you saw how differential forms neatly organizes the concepts of gradient, curl and divergence into a unifying framework. It is good to keep the conclusions of this exercise in mind.

2. Solution to selected exercises.

Part 2, Further problems.

1(c): Many of you knew that if we pick $\phi_0 = x_1dx_3 + x_1x_2dx_4$ then $d\phi_0 = \omega_3$. However, many of you simply wrote down a system of partial differential equations when you tried to find all ϕ on $\mathbb{R}^4$ for which $d\phi = \omega_3$. In fact the important point here is being able to *solve* the system you wrote down, so let me do so. Notice that if ϕ is another 1-form for which $d\phi = \omega_3$, then $d(\phi - \phi_0) = 0$ on $\mathbb{R}^4$, and since $\mathbb{R}^4$ is star-shaped (indeed simply-connectedness will do), we can invoke Poincare's lemma and conclude that ϕ differ from ϕ_0 by an exact 1-form. In other words, all ϕ for which $d\phi = \omega_3$ on $\mathbb{R}^4$ is of the form $x_1dx_3 + x_1x_2dx_4 + df$, where f is any smooth function on $\mathbb{R}^4$.

2: Many of you followed Spivak's solution, but I think it is interesting to see one using Stokes theorem too, so here it is (and it is remarkably neat). To see that ω is not exact on $\mathbb{R}^2 \setminus \{(0,0)\}$, just notice that if there were a function f on $\mathbb{R}^2 \setminus \{(0,0)\}$ for which $df = \omega$ there, then

$$\int_C \omega = \int_C df = \int_{\partial C} f = 0$$

where C is the unit circle in $\mathbb{R}^2$. But

$$\int_C \omega = \int_0^{2\pi} d\theta = 2\pi \neq 0,$$

so this leads to contradiction.

Notice that it is NOT correct in this case to identify C as the boundary of the unit disk D and say that $\int_C df = \int_D d(df) = \int_D 0 = 0$ by Stokes theorem; this is because f is not smooth (or even defined) at the origin, which is contained in D. Actually if this argument had worked, then the integral of any closed (not necessarily exact) 1-forms on the unit circle would be zero, but this is not true.

(A little aside: if you get to study topology in the future, this is basically saying that you can detect a hole in your space (namely the origin in the plane in this case) by integrating a differential form over your space. This is the most intuitive model for *cohomology* (called DeRham cohomology), and comes up a lot in the study of geometry and topology.)