MAT 218 FALL 2008
FEEDBACK ON PROBLEM SET 8

Most of you did well this week. I will discuss some useful little tricks in estimating integrals, and outline some alternative solutions to the problems.

1. Comments.

Exercises from Folland.

4.7.2. When we integrate radial functions on $\mathbb{R}^3$, one does not need the full force of the spherical coordinates system: one just needs to notice that if $F(x, y, z) = f(r)$ is a radial function, then

$$\int_{\mathbb{R}^3} F(x, y, z)dxdydz = \int_0^\infty f(r)4\pi r^2 dr$$

because the sphere of radius r has area $4\pi r^2$. Of course this would also follow if you use the spherical coordinate system (namely, writing $dxdydz$ as $r^2 \sin \phi drd\theta d\phi$), but the above formula is more strict forward (and easier to prove than the spherical coordinates formula). More generally, the following is true: if $d\sigma$ denotes the surface measure of the unit sphere in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty \int_{\theta\in\mathbb{S}^{n-1}} f(r, \theta)r^{n-1}drd\sigma(\theta).$$

Some of you have complicated arguments estimating the integrals. But if we just want to prove that a certain integral diverges, sometimes a very weak lower bound suffices: for instance, the only reason that

$$\int_{\mathbb{R}^3} \frac{dV}{1 + x^2 + y^2 + z^2}$$

diverges is that it is too big at infinity. Hence to show that it diverges, it suffices to say that the integrand is comparable to r^{-2} when $r > 1$ (meaning that $(1 + x^2 + y^2 + z^2)^{-1}/r^{-2}$ is bounded above and below by an absolute constant in this region) and that the latter is not integrable at infinity.

By the same token,

$$\int_{x^2+y^2<1} \frac{x^2 dA}{(x^2 + y^2)^2}$$

diverges: when $|y| < |x|$, x^2 is comparable to r^2, so

$$\frac{x^2}{(x^2 + y^2)^2} \simeq \frac{r^2}{r^4} = \frac{1}{r^2},$$

and the latter is not integrable in the sector $|y| < |x|$ near the origin.

5.2.3. Some of you assumed at the outset that the curve C encloses a sector of the form

$$\{(r \cos \theta, r \sin \theta) : r_1 < r < r_2, \theta_1 < \theta < \theta_2\}$$

but there is no reason assuming that this is true. The correct argument is as follows. If S is the region enclosed by a positively oriented simple closed curve C, then

$$\int_C y^3 dx + (3x - x^3)dy = \int_S 3(1 - x^2 - y^2)dxdy.$$

But the latter is an integral of a function on a region in the plane, and this integral is maximized when

$$S = \{(x, y) \in \mathbb{R}^2 \colon f(x, y) \geq 0\}.$$

Hence the original integral is maximized when C is the boundary of the unit disk in $\mathbb{R}^2$, i.e. when C is the unit circle in $\mathbb{R}^2$.

By the way, why is the fact that C is positively oriented important here? (Hint: the integral doesn't have a maximum if we allow negatively oriented curves!) Also, why is it important for C to be simple?

2. Solution to selected exercises.

Part 2, Further problems.

1: Many of you knew how to use Green's theorem to argue that if γ is any simple closed curve on the plane, and $\mathbf{F}(x_1, x_2) = (x_1, x_2)$, then

$$\int_\gamma \mathbf{F} \cdot d\mathbf{x} = \int_\gamma x_1 dx_1 + x_2 dx_2 = \int_D d(x_1 dx_1 + x_2 dx_2) = 0$$

where D is the region that γ encloses.

Here's a different argument that uses another version of Stoke's theorem: Notice that if $\mathbf{F}(x_1, x_2) = (x_1, x_2)$, then

$$\mathbf{F} \cdot d\mathbf{x} = x_1 dx_1 + x_2 dx_2 = df$$

where

$$f(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2).$$

Hence the integral of this along any (piecewise smooth) closed curve (not necessarily simple) in the plane is zero: indeed if $\gamma \colon [0, 1] \to \mathbb{R}^2$ is any smooth closed curve in the plane, then

$$\int_\gamma \mathbf{F} \cdot d\mathbf{x} = \int_\gamma df = f(\gamma(1)) - f(\gamma(0)) = 0.$$

Notice that in this argument we need not use the fact that the curve is the boundary of a region in the plane, but we need to use the additional fact that the 1-form $\mathbf{F} \cdot d\mathbf{x}$ is exact (and not only closed).