MAT 218 FALL 2008
SUMMARY ON MULTIVARIABLE INTEGRATION

Let me outline, in this article, what you'll learn in this course about multivariable integration. I will motivate these by discussing the questions that naturally arise if one ever wants to think about multivariable integration.

1. Where do we integrate?

In $\mathbb{R}^n$ there are subsets of various dimensions over which one would like to integrate. The simplest one is an n-dimensional region in $\mathbb{R}^n$, say an n-dimensional cube or ball. But there are others:

- 1-dimensional curves
- 2-dimensional surfaces

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- $(n-1)$-dimensional subsets (usually called hypersurfaces)

If we are pedantic we could also add in 0-dimensional objects, namely some discrete points in the space.

Since we will mostly be working in dimensions ≤ 3, we usually only integrate over either a n-dimensional region in $\mathbb{R}^n$, or over curves in $\mathbb{R}^n$, or over surfaces in $\mathbb{R}^3$.

To actually carry out these integrations, we would need to be able to describe these domains of integration; one useful way is given by *parametrization*.

2. What and how do we integrate?

A natural thing to integrate is a function on $\mathbb{R}^n$. In particular, the integration of the constant function 1 should be of interest, because that should give the length of a curve, the area of a surface, and the volume of a region in $\mathbb{R}^n$.

To do so, we will always need a notion of *volume element* on the domain of integration. If one is integrating a function over a region in $\mathbb{R}^n$, the volume element is simply just

$$dx_1 \dots dx_n;$$

if one is integrating a function over a parametrized curve $\gamma(t)$ in $\mathbb{R}^n$, the volume element is

$$|\gamma'(t)| dt;$$

if one is integrating a function over a parametrized surface $X(u, v)$ in $\mathbb{R}^3$, the volume element is

$$\left| \frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} \right| dudv.$$

These come out of the formula for change of variables when we integrate a multivariable function, and one needs to check, in the latter two cases, that these are well-defined (independent of the choice of the parametrizations).

There are other things, other than functions, that we would naturally want to integrate though. For instance, in physics, suppose we are given a region in $\mathbb{R}^2$ bounded by a simple closed curve, and the velocity of energy flowing out of the region at each point of the boundary curve. We would naturally want to sum these velocities over the curve, because that is how fast the total energy in the region is decreasing. But the velocity is a vector, and therefore we are naturally asked to integrate a vector field $\mathbf{v}$ over a curve C. There are a number of ways of doing it, but here the relevant way is to integrate the normal component of the vector over the curve (which is a scalar function on the curve, and which we know how to integrate!). In other words, the relevant integral now is

$$\int_C \mathbf{v} \cdot \mathbf{n} ds$$

where $\mathbf{n}$ is the outward unit normal of the curve C, and ds is the volume element of the curve. The integral can also be written as

$$\int_0^1 \mathbf{v}(t) \cdot \mathbf{n}(t)|\gamma'(t)|dt$$

if $\{\gamma(t)\}_{0\leq t\leq 1}$, is a parametrization of the curve C. A similar question arise in higher dimensions. If we have a hypersurface S in $\mathbb{R}^n$, and we have a vector $\mathbf{v}$ at each point of the hypersurface, one natural integral to look at is the integral of the normal component of the vector over the hypersurface. This is usually written as

$$\int_S \mathbf{v} \cdot \mathbf{n} dA$$

where dA is the volume element of the hypersurface S.

Alternatively, if we are given a vector field $\mathbf{v}$ over a curve C in $\mathbb{R}^2$ (actually in any $\mathbb{R}^n$ as well) then there it also makes sense to talk about *the* tangent component of a given vector to a curve, because there is just one tangent direction for a curve. It happens that sometimes we also want to integrate the tangent component of a vector over a curve. This is usually written as

$$\int_C \mathbf{v} \cdot d\mathbf{x} := \int_0^1 \mathbf{v}(t) \cdot \gamma'(t) dt$$

if $\{\gamma(t)\}_{0\leq t\leq 1}$ is a parametrization of C. This quantifies the concept of how fast the vector $\mathbf{v}$ is rotating around the curve C.

3. Fundamental theorem of calculus

The fundamental theorem of calculus, in 1-dimension, says that

$$\int_a^b f'(t)dt = f(b) - f(a)$$

for any smooth function f. It is like saying that if you integrate the derivative of a function over a 1-dimensional set, then this is the same as integrating the function over the boundary of that 1-dimensional set. There is a corresponding version in multivariables, called the Stokes theorem. It can take various forms in 2 and 3 dimensions (sometimes also called Green's theorem and Divergence theorem, respectively). It is important to understand its statement (e.g. the formulation of the boundary of a region, and the correct notions of 'derivatives' to be used in these theorems, like the curl and divergence of a vector field), understand how it is used, and its various variations. In particular, you should be able to understand and apply the following formula:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \int_S \text{curl}\mathbf{F} dA$$
$$\int_{\partial S} \mathbf{F} \cdot \mathbf{n} dA = \int_S \text{div}\mathbf{F} dV$$

4. Differential forms

Finally, the differential forms provide a unifying framework towards understanding multivariable calculus (and beyond). All the above can be formulated in terms of integration of differential forms, and that is very convenient in calculations for two reasons: first the chain rule is implicitly encoded in the formalism of pullback of differential forms, so the change of variable formula is automatic once you write everything in terms of integrals of differential forms; second the notion of derivatives of differential forms is particularly easy to work with, and it is particularly neat to state Stoke's theorem in terms of differential forms. See section 5.9 of Folland for a down-to-earth introduction to these ideas. The study of differential forms also lead to a deeper understanding of the *topology* of *manifolds*; roughly speaking, this says that one can understand the shape of objects by studying differential forms on those objects. One instance of this is Poincare's lemma. It tells you when you can always shrink a 1-dimensional loop to a point within a region without leaving the region or tearing it up. So there is cool stuff out there, and I hope you'll enjoy learning about them.