MAT 330 SPRING 2009
REVIEW SESSION 1

In the following we shall give some motivation for the study of Fourier series, indicating some of its applications. Some of these have been given in the lecture, and I shall add just a little more to put things in a more general context. We shall also have a quick review of the facts from elementary analysis that we shall frequently need throughout this course. In particular we shall discuss issues of convergence of sequences and series of functions.

1. Motivations for the study of Fourier series

The study of Fourier series shall occupy the first half of this course, and we have seen some good reasons for this in the lectures. There we have seen how Fourier series naturally arise from the solution of the (standing) wave equation and the steady-state heat equation. The crucial observation is that when you solve a partial differential equation by separation of variables, which is a particularly natural thing to do in the case of the wave equation because of the physical standing wave interpretation of its solution, one often runs into a series of sines and cosines, and the following question naturally arises:

Given an 'arbitrary' periodic function on the real line that has period 1, can we always express it in a series of the form

$$\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}?$$

We shall discuss this question in great detail in the forthcoming chapters.

I shall just try to put things in a bigger and more algebraic context by saying that a periodic function on the real line that has period 1 can naturally be identified with a function defined on the *group* $T := \mathbb{R}/\mathbb{Z}$ under addition mod 1 (or equivalently the multiplicative group of complex numbers of modulus 1; T for 1-dimensional 'torus'). (How?) The group T acts on the space of all functions on T by *translations*: if f is a function on T and $\theta \in T$, we can define the action of θ on f by sending f to f_θ, where

$$f_\theta(x) := f(x+\theta)$$

for all $x \in T$. (Here we are adopting the additive notation for the group law in $T = \mathbb{R}/\mathbb{Z}$.) Let's denote the action of $\theta \in T$ on the space of functions on T by τ_θ. The exponential functions $x \mapsto e^{2\pi i n x}$, where n is an integer, are *eigenfunctions* of τ_θ for all $\theta \in T$, and this is what really gives them their ubiquitous status. (What are the eigenvalues?) Recall that if A and B are linear operators on a vector space that *commutes* with each other, then any eigenspace of A is B-invariant. Since $\{\tau_\theta\}_{\theta \in T}$ is a family of *commuting* linear operators on the vector space of all functions on T (because T is abelian!), it is perhaps not so surprising that they admit simultaneous eigenvectors. What is surprising is that these exponentials already form a 'basis' of the space of 'all' functions on T; that is to say, 'any' function on T can be expressed in terms of linear combinations of the exponentials $\{e^{2\pi i n x}\}_{n \in \mathbb{Z}}$, and that answers our question above positively in some sense.

The above can be considered as an introduction to Fourier analysis from a representation theory point of view. Throughout this course we shall encounter two other groups, namely $\mathbb{R}^n$ and finite abelian groups, and the analogues of Fourier series in those contexts shall be our central objects of study.

2. Review of elementary analysis

Since the Fourier series is naturally a series of functions, and since we shall encounter all kinds of situations where we shall need to interchange limits with integrations and differentiations, let us review several theorems that we shall assume in this course about analysis of several real variables.

Theorem 1. *Every continuous function on a compact set is uniformly continuous.*

Theorem 2. *A sequence of real or complex numbers is convergent if and only if it is Cauchy. (This is the same as saying that $\mathbb{R}$, respectively $\mathbb{C}$, is a complete metric space.)*

Definition 1. *A series of numbers $\sum_{n=1}^{\infty} a_n$ is said to be convergent if its partial sum $\{\sum_{n=1}^{N} a_n\}_{N\in\mathbb{N}}$ converges. It is said to be absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.*

Theorem 3. *Any absolutely convergent series of real or complex numbers is convergent (but not the other way round).*

Definition 2. *A sequence of functions $\{f_n(x)\}_{n\in\mathbb{N}}$ is said to be convergent pointwisely if it is a convergent sequence of numbers at each x. It is said to converge uniformly on a set E if there is a function f on E such that $\sup_{x\in E} |f_n(x) - f(x)|$ converges to zero (as a sequence of numbers).*

Theorem 4. *The limit of a* uniformly converging *sequence of* continuous *functions is continuous.*

Theorem 5. *The limit of a* uniformly converging *sequence of (Riemann)* integrable *functions on a compact set is (Riemann) integrable, and if f_n converges uniformly to f on a compact set E, then*

$$\int_E f_n(x)dx \to \int_E f(x)dx.$$

Theorem 6. *If a sequence of differentiable functions f_n converges pointwisely to a function f on some open set and if their* derivatives *∇f_n converges* uniformly *to some function g on the same open set, then f is differentiable on that open set and $\nabla f = g$. (Have you seen the proof in $\mathbb{R}^n$?)*

Definition 3. *A series of functions is said to be uniformly convergent on a set E if its partial sums are uniformly convergent in E.*

The above theorems apply readily to uniformly convergent series of functions. What breaks down if you don't have uniform convergence?

Definition 4. *A series of functions $\sum_{n=1}^{\infty} f_n(x)$ is said to be absolutely convergent if it is an absolutely converging series of numbers for each x.*

One of the most common mistakes here is that some students tend to think that absolutely convergent series of continuous functions are continuous, but this is NOT true! (Counter-example?)

Actually absolute convergence of series of functions is rather weak, and usually does little more than telling you that the series converges pointwisely. Its main importance lies in the fact that absolutely convergent series can be rearranged freely without affecting the limit.

Note, however, that if a *Fourier series* is absolutely convergent at one point then it is uniformly convergent on the whole unit circle. (Why?)

One of the main themes of the course is to understand when and in what sense does the Fourier series of a function converge, and whether it converges to itself.

(For comparison, recall that the Taylor series of a C^∞ function may not converge even pointwisely, and even if it converges it may not converge to the original C^∞ function!)

Other convergence that we shall encounter in this course are L^2 convergence and possibly L^1 convergence: we say f_n converges to f in L^2 on a set E if

$$\int_E |f_n(x) - f(x)|^2 dx \to 0;$$

and in L^1 on E if

$$\int_E |f_n(x) - f(x)| dx \to 0.$$

Throughout the course, only Riemann integrals will be used. However, since some of you are already familiar with Lebesgue integration, and since many things are so much more convenient when you use Lebesgue integrals, you are allowed to use Lebesgue integrals (and the convergence theorems associated with them) in your problem sets, exams, etc (but please make it clear what integrals you are using unless it is clear by context). For those of you who know Lebesgue integrals, here are some theorems that you might want to use as is appropiate:

Theorem 7 (Bounded convergence theorem)**.** *If f_n is a sequence of uniformly bounded functions on a set E of finite measure and if f_n converges to f almost everywhere, then f is (Lebesgue) integrable with*

$$\int_E f_n \to \int_E f.$$

Theorem 8 (Monotone convergence theorem)**.** *If f_n is a sequence of non-negative functions on a set E and if f_n increases monotonically to a function f at almost every point of E, then*

$$\int_E f_n \to \int_E f.$$

Theorem 9 (Dominated convergence theorem)**.** *If f_n is a sequence of functions on a set E that converges to f almost everywhere, and if there exists a (Lebesgue) integrable function g on E such that $|f_n| \le g$ for all n almost everywhere, then f is (Lebesgue) integrable with*

$$\int_E f_n \to \int_E f.$$

Theorem 10 (Fubini's theorem)**.** *If $f(x,y)$ is a measurable function on $\mathbb{R}^{n+m}$ and if the iterated integral*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |f(x,y)| dy dx$$

is finite, then f is integrable over $\mathbb{R}^{n+m}$, all the following iterated integrals exist, and the following equality holds:

$$\int_{\mathbb{R}^{n+m}} f(x,y) dx dy = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f(x,y) dy dx = \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(x,y) dx dy.$$

3. Food for thought

1. Show that the series
$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2} e^{inx}$$
is absolutely and uniformly convergent on $[0, 2\pi]$.
2. More generally, show that if
$$|a_n| \leq \frac{1}{n^2}$$
for all integers n, then the series
$$\sum_{n=-\infty}^{\infty} a_n e^{inx}$$
is absolutely and uniformly convergent on $[0, 2\pi]$.