MAT 330 SPRING 2009
REVIEW SESSION 11

1. Review

The course will conclude by completing the proof of Dirichlet's theorem on primes in arithmetic progressions. In particular, we show that $L(s, \chi)$ is bounded as $s \to 1$ if χ is a non-trivial Dirichlet character mod N.

2. Further reading

There are a number of books that one can read if one is interested in more advanced Fourier analysis. They include:

- E.M. Stein and G. Weiss, Introduction to Fourier analysis on Euclidean spaces
- E.M. Stein, Singular integrals and differentiability properties of functions
- E.M. Stein, Harmonic Analysis
- Y. Katznelson, Introduction to harmonic analysis
- Y. Meyer, Wavelets and operators

The webpage of Terrence Tao (http://www.math.ucla.edu/∼tao) contains a lot of good lecture notes on Fourier and other kinds of analysis. If you want to learn more about partial differential equations, the standard references are

- D. Gilbarg and N. Trudinger, Elliptic partial differential equations of second order
- L. Hormander, The analysis of linear partial differential operators, I-IV
- L. Hormander, Lectures on nonlinear hyperbolic differential equations

However, to read most of these books, it is best if one has a very solid background of real analysis. That means one should know Lebesgue integration, Sobolev spaces, distribution theory and functional analysis. The latter three topics are covered to various extents in the books above, while a very good exposition of Lebesgue integrals is contained in

- E.M. Stein and R. Shakarchi, Real analysis

In particular, you are very encouraged to continue in the analysis sequence, MAT 331-333, if you are interested in learning more analysis.

3. Finale

To conclude the course, let me list some questions that you may want to think about when you review the course:

1. State and prove the relation between the Fourier transform and the following operations, both on $\mathbb{R}^d$ and on the unit circle (if applicable).
   (a) Differentiation
   (b) Multiplication by polynomials
   (c) Translation
   (d) Modulation

   (e) Rotation
2. State precisely and prove:
   (a) Parseval's identity on the unit circle
   (b) Phancherel's theorem on the real line or on finite abelian groups
   (c) the Fourier inversion formulae
   (d) Poisson summation formula
   (e) Heisenberg's uncertainty principle
   (f) Weyl's criterion on equidistribution
   (g) Isoperimetric inequality in the plane
   (h) Peter-Weyl theorem on finite abelian groups
3. Define, in two equivalent ways, the L-function assoicated with a Dirichlet character, and explain some of their simple properties.
4. Explain the relevance of the non-vanishing of $L(s, \chi)$, where χ is a non-trivial Dirichlet character mod N, to Dirichlet's theorem on primes in arithmetic progressions.
5. State precisely an existence and uniqueness theorem for the following partial differential equations:
   (a) Laplace equation
   (b) Heat equation

   You should consider a boundary value problem or an initial value problem as appropiate. Do this both on the real line and the unit circle.
6. Give a physical representation of a solution of the initial value problem for the wave equation in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$. Give also the Fourier representation. What are each of these representations good for?
7. Describe the decay of the canonical solution of the heat equation when the initial value is a Schwartz function. Do this also for the Laplace equation on the upper half-plane, when the boundary value is Schwartz.
8. Define the energy of a solution to the wave equation. Show that the energy is conserved if the solution is smooth and compactly supported in space.
9. State and prove a statement that explains the phenomenon of 'finite speed of propagation'.
10. Define the Radon transform in $\mathbb{R}^2$ and $\mathbb{R}^3$. Explain its relation to the Fourier transform, and how it is inverted in $\mathbb{R}^3$.
11. Explain the relation between Bessel functions and the Fourier transform.

Apart from the above, I hope you have also gained experience making various estimates, and enjoyed the course as a whole.