MAT 330 SPRING 2009
REVIEW SESSION 2

In the following we shall review two major ideas that have been covered in class. The first one concerns how well a function is determined by its Fourier series. The second one relates the study of pointwise convergence and summability of Fourier series to convolutions with bad and good kernels.

A little extra about convolutions that arise in number theory is included for those interested, and some hints to the harder problems follow.

1. Review

We have seen that if the Fourier coefficients of a Riemann integrable function are all zero, then it is zero at its points of continuity. Moreover, if a Fourier series converges absolutely at one point (for instance when the function is C^2), then it converges uniformly on the unit circle, and the function is equal to its Fourier series at its points of continuity.

We have also seen that the partial sums of a Fourier series of a function can be obtained by convolution with the Dirichlet kernel. The Cesaro and Abel means can be obtained by convolutions with the Fejer kernel and the Poisson kernel respectively. These last two kernels are good, and give us summabilities of Fourier series of a function at its points of continuity (uniform summabilities if the function is uniformly continuous to begin with). The Dirichlet kernel, however, is not a good kernel and does not give any good result so far. Next week we will see how the Abel means of the Fourier series of a function solves the Dirichlet problem on the unit disk in $\mathbb{R}^2$.

2. A little extra

You may have seen some versions of convolutions before. A complex-valued function defined on the positive integers is sometimes called an *arithmetic function*. If f and g are two arithmetic functions then we can define their *convolution* to be the arithmetic function defined by

$$f * g(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

It is easy to verify that this makes the set of all arithmetic functions a commutative ring with unit, and the famous Mobius μ function is just the inverse of the function on $\mathbb{N}$ that is identically 1. The Mobius inversion formula then becomes particularly transparent.

You will see more on arithmetic functions if you take MAT 331, Complex Analysis.

3. Hints to Problem set 2

The problems are taken from Chapter 2 of Stein and Shakarchi.

6(d). You need to show why the Fourier series converges to the original function (say by quoting an appropiate theorem from the book).

7(a). This is just a discrete version of integration by parts. (What are the boundary terms and where are the derivatives?) So one can adapt the proof of integration by parts here.

7(b). Cauchy's criterion is a good way to show convergence when you don't know what the limit should be.

12. Compute $\sigma_n - s_n$ and show that it converges to 0 as $n \to \infty$. You may want to exhibit $\sigma_n - s_n$ as an explicit sum, and see why (intuitively first) that the sum should be small.

13(a). Let $\sum c_n$ converge to s. To show Abel convergence of this series, we have to consider the series

$$\sum c_n r^n$$

for $r < 1$. Since we expect it to converge back to s, one natural thing to do is to relate the above sum to s_n, the partial sums of $\sum c_n$. Since $s_n = c_1 + \cdots + c_n$, it is natural to use summation by parts here.