MAT 330 SPRING 2009
REVIEW SESSION 3

By now we have seen several ways in which a Fourier series could converge. Each happens under a different set of hypothesis. We summarize this below. Some hints to the homework problems are given.

1. Review

We have discussed four kinds of convergence, namely mean-square convergence, summability, pointwise convergence and absolute convergence. Roughly speaking, the first two are rather weak and hold under fairly general assumptions. The last two are stronger and only hold for sufficiently smooth functions.

At this early stage we need a function to be Riemann integrable (and in particular, bounded) on the unit circle to talk about its Fourier series. So we shall assume all functions to be periodic and Riemann integrable in this section.

1.1. **Mean-square convergence.**

The exponentials $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ form a complete orthonormal set under the L^2 inner product on $[0,1]$, and hence (the symmetric partial sums of) the Fourier series of every Riemann integrable function $f(x)$ converges in L^2 norm to $f(x)$; in fact $f(x) \mapsto \{\hat{f}(n)\}$ maps the space of Riemann integrable functions isometrically into the complete inner product space l^2, and the Parseval's identity

$$\int_0^1 |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

holds.

1.2. **Summability.**

A function $f(x)$ is both Cesaro and Abel summable at a point x_0 to the function value at x_0 if it is continuous at x_0. In particular, if the Fourier series of $f(x)$ converges at a point x_0 and if f is continuous at x_0, then the limit of (the symmetric partial sums of) the Fourier series of f at x_0 is equal to the function value $f(x_0)$. (Compare with the Taylor series of a C^∞ function, which may converge to something other than the function value even when it is given that it converges.) Also, a function is uniformly Cesaro and Abel summable on a set E if it is uniformly continuous on E. The Abel sum of a continuous function gives the solution to the Dirichlet problem on the unit disc.

1.3. **Pointwise convergence.**

If $f(x)$ is differentiable (or just Lipschitz) at a point x_0 then (the symmetric partial sums of) its Fourier series converges to $f(x_0)$ at x_0. This follows from the Riemann-Lebesgue lemma, which states that the n-th Fourier coefficients of an integrable function tends to 0 as n tends to infinity. This establishes the localization principle of Riemann: if $f(x) = g(x)$ in an open interval containing a point x_0, then the Fourier series of the two functions either both converge to the same value at x_0 or both diverge at x_0.

1.4. **Absolute convergence.** If the Fourier series of a function converges absolutely, then it converges uniformly. This happens, for instance, if the function is C^2 on the whole unit circle because then the Fourier coefficients decay like $1/n^2$. (This also happens for functions that are globally in C^α for some $\alpha > 1/2$; this can be shown by estimating $\sum_{2^k \leq |n| \leq 2^{k+1}} |\hat{f}(n)|^2$ using the Parseval's formula. See Exercises 14 and 16 in Chapter 3.)

1.5. **Pointwise Divergence.** We have also seen an example of pointwise divergence: there is a continuous periodic function on $[0, 1]$ whose Fourier series diverges at a point.

Finally, all the above are about summability or convergence of Fourier series. There are, however, examples of trigonometric series that converges everywhere on the unit circle without being the Fourier series of any (Riemann integrable) functions. (See Exercise 7 in Chapter 3.)

So to conclude, we have seen the following function spaces, in decreasing order of generality: the spaces of Riemann integrable functions, continuous functions, Holder continuous functions, Lipschitz functions, differentiable functions, C^1 functions and C^2 functions. All except the first one are characterized by local conditions, and we may require them to hold either globally or locally. The space of continuous functions, while good enough for summability, is usually too weak for convergence of Fourier series; the space of (locally) differentiable (or just Lipschitz) functions is good for pointwise convergence, while the space of globally C^2 (or just C^α, $\alpha > 1/2$) is good for absolute (and hence, in this case, uniform) convergence.

2. A little extra

The space of Riemann integrable functions on the unit circle is not complete under the L^2 inner product. Hence it is natural to seek a completion of this space, namely the smallest complete inner product space containing it. This is usually called the space of L^2 functions; it is an interesting space with many nice properties, and also a prototype of complete inner product spaces. For instance, this is isometric to the complete inner product space l^2 under the map $f \mapsto \{\hat{f}(n)\}_{n\in\mathbb{Z}}$. You will learn more about it when you study real analysis.

We can also consider the completion of the space of Riemann integrable functions under the L^p norm, namely

$$\|f\|_{L^p} = \left(\int |f|^p dx \right)^{\frac{1}{p}},$$

where $1 \leq p < \infty$. This is usually called the space of L^p functions. The Riemann-Lebesgue lemma still holds for all L^1 functions on the unit circle. It is then easy to see that if $f(x)$ satisfy a Dini condition at x_0, i.e.

$$\int_{-1}^{1} \frac{|f(x_0 + t) - f(x_0)|}{t} dt < \infty$$

(this holds for instance if f is C^α at x_0 for some $\alpha > 0$), then the Fourier series of f converges at x_0, generalizing our previous sufficient condition for pointwise convergence of Fourier series.

Concerning the convergence of Fourier series, other than the results we have seen, two more important results are often mentioned. The first one is a generalization of mean-square convergence, and says that if $1 < p < \infty$ and $f \in L^p$, then the partial sums of the Fourier series of f converges to f in L^p, i.e.

$$\int |S_n(f)(x) - f(x)|^p dx \to 0$$

as $n \to \infty$. This is one of the landmarks in harmonic analysis in the early 20th century, and the result is rather delicate; in fact it is false when $p = 1$ or $p = \infty$. Another concerns the pointwise convergence of Fourier series of an L^2 function, and says that the Fourier series of an L^2 function converges pointwisely except possibly on a set of measure zero. This is an even deeper result (due to Carleson in 1966); in fact while the analogous statement for L^p is true for $1 < p < \infty$ (this is also hard), the analogous statement for L^1 fails so miserably that there exists an L^1 function on the unit circle whose Fourier series diverges at every point (Kolomorgov, 1920s).

3. Hints to Problem set 3

The exercises are taken from Chapters 2 and 3 of Stein and Shakarchi.

2.19. The formula is simpler if you first extend f to be odd and write the solution in the form

$$\int_{-1}^{1} f(t)Q(x-t,y)dt$$

for a suitable kernel Q. Just work formally here since no condition about f has been given to guarantee convergence etc.

3.2 Given $\varepsilon > 0$, there exists N such that

$$\|A_n - A_m\| < \varepsilon$$

for all $n, m \geq N$. In particular,

$$\|A_n - A_N\| < \varepsilon$$

for all $n \geq N$. Hence if K is chosen such that

$$\sum_{k=K+1}^{\infty} |a_{N,k}|^2 < \varepsilon^2$$

(why does such K exist?), then

$$\sum_{k=K+1}^{\infty} |a_{n,k}|^2 < (2\varepsilon)^2$$

for all $n \geq N$ (why?). It follows that $B \in l^2$ (why?), and it is now easy to show that $A_n \to B$ in l^2 by observing that

$$\|A_n - B\|^2 \leq \sum_{k=1}^{K} |a_{n,k} - b_k|^2 + 2\sum_{k=K+1}^{\infty} |a_{n,k}|^2 + 2\sum_{k=K+1}^{\infty} |b_k|^2.$$

(Why does the last inequality hold?)

3.3 Draw pictures of what you guess f_k might be.

3.10. Since u is just known to be C^1 and no better, the best way to proceed is to use the explicit solution formula

$$u(x,t) = F(x+ct) + G(x-ct)$$

in Chapter 1 and argue that $E(t) = E(0)$. If more differentiability were known, however, then the conservation of energy can also deduced by differentiating $E(t)$ and using the wave equation to argue that $E'(t) = 0$ for all t. It might also help to renormalize the functions so that $c = 1$.