MAT 330 SPRING 2009
REVIEW SESSION 4

1. Review

This week we have seen how to derive the isoperimetric inequalities for areas enclosed by plane curves using the Parseval's identity for Fourier series. We have also seen a number of characterizations of the equidistribution of a sequence of numbers on the unit interval, and shown that if γ is an irrational real number then the fractional parts of $n\gamma$ is equidistributed on $[0,1]$. We concluded with an example of a continuous (in fact C^α) function that is nowhere differentiable on $[0,1]$.

2. A little extra

We have seen in the problem set that the isoperimetric inequalities for areas enclosed by plane cures is equivalent to the Wirtinger's inequality, which says that

$$\int_0^{2\pi} |f(x)|^2 dx \leq \int_0^{2\pi} |f'(x)|^2 dx$$

for all C^1 functions f on $[0,2\pi]$ whose integral $\int_0^{2\pi} f(x)dx = 0$. It might be of interest to know that there is another analytic inequality that is equivalent to this, namely the Sobolev inequality: it says that for compactly supported smooth functions u defined on the plane, we have

$$\int_{\mathbb{R}^2} |u(x,y)|^2 dxdy \leq \frac{1}{4\pi} \left(\int_{\mathbb{R}^2} |\nabla u(x,y)| dxdy \right)^2 .$$

Note the appearance of the constant 4π on the right hand side of the Sobolev inequality; it is of no coincidence that the same constant appears in the isoperimetric inequality. Note also that both the Sobolev and Wirtinger's inequalities controls the size of a function by its derivatives, and the isoperimetric inequality shares the same feature in that it controls the volume of an object by the size of its boundary. (Recall that taking boundaries is roughly speaking the same as taking derivatives, as Stoke's theorem in multivariable calculus reveal.) There are also interesting generalizations of the Sobolev inequality and the isoperimetric inequalities to higher dimensions.

3. Hints to Problem set 4

The exercises are taken from Chapters 3 and 4 of Stein and Shakarchi.

3.6. Show that if f is such a function, then it is unbounded because its Abel means at 0 is unbounded as $r \to 1$.

3.12. Remember to show that

$$\lim_{y\to\infty} \int_0^y \frac{\sin x}{x} dx$$

converges as $y \to \infty$.

3.14. Use

$$\left(\sum |\hat{f}(n)| \right)^2 \leq \left(\sum |n\hat{f}(n)|^2 \right) \left(\sum \frac{1}{n^2} \right).$$

4.4. To show that isoperimetric inequality implies Wirtinger, without loss of generality assume f is real-valued. Then say $f(s) = y(s)$, and let $x(s)$ be such that $x'(s) = -y(s)$. Check that x is C^1 and periodic of period 2π so that $(x(s), y(s))$ parametrizes a C^1 closed curve, and note that

$$\int_0^{2\pi} y'(s)^2 - y(s)^2 ds = \int_0^{2\pi} (x'(s)^2 + y'(s)^2) + 2\int_0^{2\pi} x'(s)y(s)ds.$$

Conclude by noticing that $A = -\int_0^{2\pi} x'(s)y(s)ds$, and that

$$\int_0^{2\pi} (x'(s)^2 + y'(s)^2)ds \geq \frac{L^2}{2\pi},$$

where A and L are the area of the region enclosed, and the length of the boundary of the region enclosed, respectively.

4.5 Note that

$$\left(\frac{1-\sqrt{5}}{2}\right)^n \to 0.$$