MAT 330 SPRING 2009
REVIEW SESSION 5

1. Review

This week we concluded our study of Fourier series with some further applications towards the heat equation on the unit circle. We then began to study Fourier transform on the real line, and saw many properties of the Fourier transform analogous to those of the Fourier series of a periodic function. In the following I shall highlight the similarities and the differences of the two theories, and hopefully this could put things in a slightly different perspective.

The central question in both the study of Fourier series and Fourier transform is the following: How well can a function be reconstructed from its Fourier decomposition? Several aspects of the theories are really parallel of each other:

(1) Decay of Fourier coefficients vs smoothness of the functions
(2) Behaviour of Fourier coefficients under translation, modulation and dilation of functions
(3) Convolutions and their close relation to Fourier transform, as illustrated by the formula $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$
(4) Representation of a bounded (uniformly) continuous function as the (uniform) limit of its convolution with a good kernel
(5) Parseval or Phancherel's formula

There are, however, also significant differences in our considerations of the two theories:

(1) The most important one is now that we are always working with Schwartz functions, or at least (continuous) functions which together with its Fourier transform are of moderate decrease. This guarantees automatically the absolute convergence of the Fourier inversion integral
$$\int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi;$$
in other words, we are only dealing with analogues of absolutely convergent Fourier series on the real line. Therefore the Fourier inversion formula is easy to derive and holds pointwise; we never had to write 'partial integrals'
$$\int_{-N}^{N} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi$$
as convolutions of f against some kernels D_N and study the behaviour of such convolutions (although we could in principle, and that leads to interesting mathematics when f is allowed to be more general functions).
(2) One important formula in this theory is the multiplication formula, which states that for Schwartz functions f and g,
$$\int_{-\infty}^{\infty} f(x) \cdot \widehat{g}(x) dx = \int_{-\infty}^{\infty} \widehat{f}(\xi) \cdot g(\xi) d\xi.$$
This can be proved by interchanging the order of integration, and can be justified by some version of the Fubini's theorem.

(3) There is just one important good kernel in the theory of Fourier transform, namely (the scaled versions of) the Gaussian. The key feature of the Gaussian is that it is its own Fourier transform:

$$\widehat{e^{-\pi x^2}} = e^{-\pi \xi^2}.$$

Since its Fourier transform can be computed explicitly, it is tempting to plug it into the multiplication formula above and see what we can get.

(4) Let f be a Schwartz function. If we now take g to be a scaled version of the Gaussian, say

$$g(\xi) = e^{-\pi \delta \xi^2},$$

and apply the multiplication formula, we then get

$$\int_{-\infty}^{\infty} f(x) \delta^{-\frac{1}{2}} e^{-\frac{\pi x^2}{\delta}} dx = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{-\pi \delta \xi^2} d\xi$$

for all $\delta > 0$. Letting $\delta \to 0$ we get

$$f(0) = \int_{-\infty}^{\infty} \widehat{f}(\xi) d\xi,$$

which is the Fourier inversion formula for Schwartz functions at 0.

(5) By translating the above identity (considering $y \mapsto f(x+y)$ for a given x), we get the Fourier inversion formula for Schwartz functions f at any point x.

(6) The role of orthogonality is apparently suppressed in the Phancherel's identity, because now that we have the Fourier inversion formula already the Phancherel's identity (at least for Schwartz functions) is an easy consequence of that. Nevertheless, the orthogonality is still implicitly there, and this is an important aspect of the theory. (By the way, it may be of interest to know that the Phancherel's identity can also be derived using the multiplication formula.)

2. Hints to Problem set 5

One theme in this course (indeed any analysis course) is the prevalence of estimates. This week in the problem set we shall deal with a number of them. If you are not familiar with how these things go, or if you don't understand why you would want to do this and that in doing these estimates, it may be a good time now to sit down and think for yourself why the estimates have to be done this way.

A few good places to look up some sample estimates are:

(1) Corollary 2.2.4, p.43
(2) Proposition 2.3.1(v), p.46-47
(3) Theorem 2.4.1, p.49
(4) Theorem 3.1.1, p.79
(5) Theorem 3.2.1, p.82
(6) Estimates on p.84
(7) Lemma 4.2.2, p.109
(8) Lemma 4.3.2, p.116
(9) Proposition 5.1.1, p.133
(10) Proposition 5.1.2, p.136
(11) Corollary 5.1.7, p.140

The exercises are taken from Chapters 4 and 5 of Stein and Shakarchi.

4.7. Approximate f uniformly by step functions.

4.10(a). Approximate f uniformly by trigonometric polynomials.

4.10(b). What identity should you use when the integral of a function squared comes up?

4.11. Use Parseval, and split the sum into two parts. Deal with the two parts of the sum separately. (Think intuitively: why should each of them be small?) Also compare with 4.10(b).

4.13(b) You may wonder why one could come up with the idea of using $x^2 \leq C\sin^2(\pi x)$ if it were not given to you as a hint. However, remember multiplying by x on the function side basically corresponds to 'differentiation' on the frequency side. So it is natural to try to come up with some terms that involve the difference of the Fourier coefficients of the heat kernel.