MAT 330 SPRING 2009
REVIEW SESSION 6

1. Review

We have seen how the Fourier transform can be used to solve the heat equation on the real line and the Dirichlet problem on the upper half plane. We have also seen a simple application of the Poisson summation formula, which enabled us to prove that the heat kernel on the unit circle is a good kernel.

2. A little extra

We have seen that a harmonic function on the plane satisfies the mean-value property. In fact the same is true in higher dimensions: if u is a C^2 function that satisfies $\Delta u = 0$ in a neighborhood of a closed ball $B(x,r) \subset \mathbb{R}^d$, then

$$u(x) = \frac{1}{|S(x,r)|} \int_{S(x,r)} u(y) d\sigma(y) \tag{1}$$

where $S(x,r)$ is the sphere of radius r centered at x, $d\sigma$ is the surface carried measure on $S(x,r)$ and $|S(x,r)|$ is the area of the sphere $S(x,r)$. In other words, the value of u at x is the average of the values of u on $S(x,r)$.

Integrating over r, we also get

$$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y) dy \tag{2}$$

under the same conditions, where now $|B(x,r)|$ denotes the volume of the ball.

What is remarkable is that the above implications are reversible: if u is a continuous function say on the whole $\mathbb{R}^d$, such that (2) holds for all $x \in \mathbb{R}^d$ and $r > 0$, then u is C^2 and $\Delta u = 0$ on $\mathbb{R}^d$. The same conclusion also holds if we assume that (1) holds for all $x \in \mathbb{R}^d$ and $r > 0$.

Hence harmonic functions are characterized by the fact that at each point its value is the average of its values in all nearby spheres or balls; this makes sense since they are the steady state solutions of the heat equation, and heat only stops flowing when the temperature at each point is the average of the temperature of its nearby points.

It may be useful to remember that harmonic functions on $\mathbb{R}$ are just linear functions.

3. Hints to Problem set 6

The exercises are taken from Chapters 4 and 5 of Stein and Shakarchi.

4.8. In estimating
$$\int_1^N e^{2\pi i b x^\sigma} dx$$
change variable $y = x^\sigma$ and integrate by parts.

5.9. Remember to verify all three properties of good kernels.

5.14. You should state why the Poisson summation formula is applicable. (Same for Exercise 5.15(a).)

5.15(b). For $0 < x < 1$, integrate both sides of (a) from $\frac{1}{2}$ to x against $d\alpha$. To evaluate the integral of the sum on the left hand side, check that the sum converges uniformly in α over the interval of integration. This allows you to exchange the infinite sum with the integral. Then one can integrate term by term, and when symmetric partial sums are taken on the left hand side, the terms involving evaluation at $\alpha = \frac{1}{2}$ cancel out massively. The general case now follows by periodicity (why?).

5.16 Note that if

$$D_N(x) = \sum_{|n| \leq N} e^{2\pi i n x}$$

is the ordinary Dirichlet kernel for periodic functions of period 1, then the modified Dirichlet kernels satisfy

$$D_N^* = \frac{1}{2}(D_N + D_{N-1}).$$

Using the explicit expression for D_N, this could be easily seen to be equal to the sum

$$\sum_{n=-\infty}^{\infty} \mathcal{D}_N(x+n)$$

upon invoking Exercise 5.15(b).