MAT 330 SPRING 2009
REVIEW SESSION 7

1. Review

We have seen the Heisenberg's uncertainty principle, and then moved on to study Fourier transform in $\mathbb{R}^d$, $d \geq 2$. Many one dimensional formula have analogues in higher dimensions, but genuinely new phenomena also occur in dimensions ≥ 2. This will become more evident when we study the solution to the wave equation in $\mathbb{R} \times \mathbb{R}^d$.

2. A little extra

The following is a more general version of the Heisenberg uncertainty principle in functional analysis. Suppose A and B are self-adjoint (or anti-self-adjoint) operators on an inner product space, for which the *commutator* $[A, B] := AB - BA$ is the identity operator. Then

$$\|\psi\|^2 \leq 2\|A\psi\|\|B\psi\|$$

for all ψ in the inner product space, where $\|\cdot\|$ denotes the norm of the inner product space.

The proof of this statement is the same as that of the version in the book. There we are just applying the above principle to the inner product space $\mathcal{S}(\mathbb{R})$, where the operators are $A\psi = \frac{d}{dx}\psi$ and $B\psi = x\psi$.

By the way, the operator $\frac{d}{dx}$ is not self-adjoint, while in quantum mechanics it is postulated that all observable quantities are represented by self-adjoint operators. Thus physicists tend to consider $i\frac{d}{dx}$ instead, which is self-adjoint under the inner product

$$(\psi, \phi) = \int \psi\overline{\phi}.$$

3. Hints to Problem set 7

The exercises are taken from Chapters 5 and 6 of Stein and Shakarchi.

5.18. It is a useful principle that one can express the power function x^{-s} ($s > 0$ fixed) as a weighted average of the exponentials e^{-xt}. One instance of this is the fact that

$$x^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt} t^{s-1} dt, \quad x > 0.$$

In this exercise we need a variant of this, namely

$$x^{-s} = \frac{1}{\pi^{-\frac{s}{2}}\Gamma(s/2)} \int_0^\infty e^{-\pi x^2 t} t^{\frac{s}{2}-1} dt, \quad x > 0.$$

5.19(b). The first formula only holds for $0 < |t| < 1$ (why? check that the radius of convergence of the power series on the right hand side is 1). To show this, it is helpful to remember the identity

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1.$$

(This is equality is evident if we begin from the right hand side, but very often we need to be able to use this to express $(1+x)^{-1}$ as a power series.)

6.5. Factorize the positive definite matrix A as B^tB, and make a change of variable $y = Bx$.