# MAT 330 SPRING 2009
# REVIEW SESSION 8

## 1. Review

This week we have completed our study of Fourier analysis on $\mathbb{R}^d$. In particular, we have seen how radial functions are transformed under the Fourier transform, and the relation of the Fourier transform of the surface measure of the unit sphere with the Bessel function. These considerations also arose in the study of the wave equation in $d+1$ dimensions, and we have seen explicit solution formula for the wave equation both using spherical averages and using the Fourier transform. The concepts of light cones and finite speeds of propagation were introduced, and the difference between odd dimensions and even dimensions were emphasized. Finally we studied the Radon transform in $\mathbb{R}^3$, and saw how that could be inverted.

## 2. A little extra

If $T$ is a linear operator that sends a (say Schwartz) function $g(x)$ on $\mathbb{R}^d$ to a solution of the wave equation

$$\begin{cases} \partial_t^2 u = \Delta u \\ u(x,0) = 0 \\ \partial_t u(x,0) = g(x) \end{cases}$$

then $T$ is said to be the fundamental solution of the wave equation. This is because from $T$ we can build the solutions of all other related problems in wave equations: for instance $\partial_t(Tf)(x,t)$ is then a solution of

$$\begin{cases} \partial_t^2 u = \Delta u \\ u(x,0) = f(x) \\ \partial_t u(x,0) = 0 \end{cases}$$

and we can then solve the general initial value problem

$$\begin{cases} \partial_t^2 u = \Delta u \\ u(x,0) = f(x) \\ \partial_t u(x,0) = g(x) \end{cases}$$

by superposition. More generally, one can even solve

$$\begin{cases} \partial_t^2 u - \Delta u = K(x,t) \\ u(x,0) = f(x) \\ \partial_t u(x,0) = g(x) \end{cases}$$

using the above fundamental solution $T$, using the *Duhamel's formula*. The idea is to think of $u(x,t)$ as a map from time $t$ to the space of functions of $x$, and think of the wave equation as an ordinary differential equation in $t$ then.

## 3. Hints to Problem set 8

The exercises are taken from Chapters 5 and 6 of Stein and Shakarchi.

5.20. This question is about sampling theory in Fourier analysis. The crucial observation is that when a function $f$ is compactly supported, say on the unit interval and say $f$ is also smooth, then there are two ways of taking the Fourier transform of $f$, namely as a function defined on the unit circle or as a function defined on $\mathbb{R}$; these two ways agree on the integers, and we shall denote both by $\hat{f}$. Now the function $f$ can be reconstructed from $\hat{f}$ in two ways: either

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$$

or

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

(both converges absolutely because $f$ is Schwartz). This says to reconstruct $f$, all that we need to know is only $\hat{f}(n)$ at the integers, and the rest is determined by that (note how we took advantage of the fact that we are working with a compactly supported function). This is the crucial observation behind sampling theory.

In part (a), note that $\hat{f}$ is supported on $[-1/2, 1/2]$. Applying the above to $\hat{f}$ in place of $f$, we get

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi}$$

with the sum converging uniformly, for $\xi \in [-1/2, 1/2]$. Fourier inverting this, we get the desired formula.

In part (b), we should think of $\hat{f}$ as a function supported on $[-\lambda/2, \lambda/2]$ and expand it in Fourier series; then

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda} f\left(\frac{n}{\lambda}\right) e^{-2\pi i n \xi/\lambda}$$

for all $\xi \in [-\lambda/2, \lambda/2]$. In fact if $\chi$ is the continuous function supported in $[-\lambda/2, \lambda/2]$ ($\lambda > 1$), identically equal to 1 on $[-1/2, 1/2]$ and linear in between $[-\lambda/2, -1/2] \cup [1/2, \lambda/2]$, then we can insert $\chi$ into the above expression and still obtain

$$\hat{f}(\xi) = \chi(\xi) \sum_{n=-\infty}^{\infty} \frac{1}{\lambda} f\left(\frac{n}{\lambda}\right) e^{-2\pi i n \xi/\lambda}$$

for all $\xi \in [-\lambda/2, \lambda/2]$.

Finally in part (c), one just needs to apply both the Parseval's identity (for functions on the unit circle) and Phancherel's formula (for functions on the real line) correctly.

6.7. To show infinite differentiability in $t$, note that it is easier to write $u$ as

$$u(x,t) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{-4\pi^2 t |\xi|^2} e^{2\pi i x \cdot \xi} d\xi$$

(Note that this formula holds at every $x$ and $t > 0$.) Then for fixed $x$ and $t > 0$, one can write

(1)
$$\begin{aligned}&\left|\frac{u(x,t+h)-u(x,t)}{t} - \int_{\mathbb{R}^d} \hat{f}(\xi)\frac{d}{dt}e^{-4\pi^2 t|\xi|^2}e^{2\pi i x\cdot\xi}d\xi\right|\\ =&\left|\int_{\mathbb{R}^d}\hat{f}(\xi)\left(\frac{e^{-4\pi^2(t+h)|\xi|^2}-e^{-4\pi^2 t|\xi|^2}}{h} - \frac{d}{dt}e^{-4\pi^2 t|\xi|^2}\right)e^{2\pi i x\cdot\xi}d\xi\right|,\end{aligned}$$

split the integral into integral over $|\xi| > R$ and $|\xi| \leq R$ for a suitable $R$ (the choice of $R$ could depend on both $f$, $x$ and $t$ now because we are fixing these for the moment) and argue that if $R$ is sufficiently large then the first integral is smaller than $\varepsilon$ (why?). Now pick $h$ to be sufficiently small (depending on $R$ as well). Then the second integral is also less than $\varepsilon$ (why?). This proves differentiability at $(x,t)$, and that the derivative is also of the form

$$\int_{\mathbb{R}^d}\hat{g}(\xi)e^{-4\pi^2 t|\xi|^2}e^{2\pi i x\cdot\xi}d\xi$$

where $g$ is Schwartz. The same argument thus shows that $u(x,t)$ is infinitely differentiable in $t$. It is not hard to show by the same argument that $u(x,t)$ is infinitely differentiable in both $x$ and $t$.

Note that although the integrand in (1) converges uniformly to 0, this is not enough to guarantee the convergence of that integral to 0. This is because we are integrating over a region of infinite volume.

6.8 To compute the inverse Fourier transform of $e^{-2\pi|\xi|}$ in $\mathbb{R}^d$, note that first of all it is easy when $d = 1$. There by splitting the integral into integral over the positive and negative real axis one gets

(2)
$$\int_{-\infty}^{\infty} e^{-2\pi|\xi|}e^{2\pi i x\xi}d\xi = \frac{1}{\pi(1+x^2)}.$$

Now when $d > 1$, to compute

$$\int_{\mathbb{R}^d} e^{-2\pi|\xi|}e^{2\pi i x\cdot\xi}d\xi,$$

we try to express $e^{-2\pi|\xi|}$ as a weighted average of scaled versions of the Gaussians $e^{-\pi|\xi|^2/t}$, whose inverse Fourier transform is easy to compute; this roughly amounts to replacing $|\xi|$ by $|\xi|^2$, and we proceed as follows. First we take the equation (2) and write, for $\beta \in \mathbb{R}$,

$$e^{-2\pi|\beta|} = \int_{-\infty}^{\infty}\frac{1}{\pi(1+\alpha^2)}e^{-2\pi i\alpha\beta}d\alpha;$$

then observe that

$$\frac{1}{\pi(1+\alpha^2)} = \int_0^{\infty} e^{-\pi(1+\alpha^2)t}dt.$$

Combining the two, and using Fubini's theorem, we get

$$e^{-2\pi|\beta|} = \int_0^{\infty} e^{-\pi t}\int_{-\infty}^{\infty} e^{-\pi\alpha^2 t}e^{-2\pi i\alpha\beta}d\alpha dt.$$

The inner integral is just the Fourier transform of the Gaussian, and we get

$$e^{-2\pi|\beta|} = \int_0^{\infty} e^{-\pi t}e^{-\pi\beta^2/t}\frac{dt}{t^{1/2}}.$$

For $\xi \in \mathbb{R}^d$, if we take $\beta = |\xi|$, then

$$e^{-2\pi|\xi|} = \int_0^{\infty} e^{-\pi t}e^{-\pi|\xi|^2/t}\frac{dt}{t^{1/2}}.$$

Taking inverse Fourier transform of both sides now, and using the Gamma function, we get an explicit formula of the inverse Fourier transform of $e^{-2\pi|\xi|}$.