MAT 330 SPRING 2009
REVIEW SESSION 9

1. Review

This week we have studied Fourier analysis on finite abelian groups. The important notion of characters was introduced, and the important theorem is that the characters of a finite abelian group G is an orthonormal basis of the space of all functions on G. We also discussed the Fourier transform on a finite abelian group, and how that leads to a notion of Fourier series on a finite abelian group.

2. A little extra

In the proof that the characters of a finite abelian group span the space of functions on that group, we have seen very clearly that the characters of a finite abelian group are just eigenfunctions of the translation operator. This suggests that the study of Fourier analysis on any group should be closely related to the translation structure of the group. For instance, the exponential functions $e^{2\pi inx}$ that arise in the study of Fourier analysis on the unit circle are just eigenfunctions of translations.

It is also intructive at this point to point out the roles of Laplacian in the study of Fourier analysis. Basically it plays two roles. First of all, on $\mathbb{R}^d$, it is the unique differential operator D that satisfies

$$\begin{cases} D(f(x+a)) = (Df)(x+a) & \text{for all } a \in \mathbb{R}^d \\ D(f(Rx)) = (Df)(Rx) & \text{for all orthogonal maps } R \\ D(f(\lambda x)) = \lambda^2 (Df)(\lambda x) & \text{for all } \lambda > 0. \end{cases}$$

In other words, it is the unique translation and rotation invariant operator that is homogeneous of degree 2 under dilations. Secondly, and this ties to the discussion about eigenfunctions of translations above, let's observe that eigenfunctions of the translation operator are also eigenfunctions of the Laplacian. This is because the Laplacian involves an infinitesimal translation on $\mathbb{R}^d$. This observation is crucial when we want to study Fourier analysis on some more general spaces, say on a Riemannian manifold M where there is no group structure, but where the Laplacian still makes sense; there we turn to study eigenfunctions of the Laplacian instead, and they often give us important information about the underlying manifold. Since harmonic functions are just functions in the kernel of the Laplace operator, and since the Laplace operator captures so much about Fourier analysis, the study of Fourier analysis is also sometimes called harmonic analysis.

3. Discussion of the problem set

In the problem set we have come across the dual or adjoint of the Radon transform. I think it may be a good idea if I explain a little bit more what the adjoint operator is in general. To fix the ideas, suppose we are in $\mathbb{C}^n$, and we see that as a Hilbert space with inner product

$$(z, w) = \sum_{j=1}^{n} z_j \overline{w_j}$$

and associated norm $\|z\| = \sqrt{(z,z)}$. A map $T\colon \mathbb{C}^n \to \mathbb{C}$ is called a linear functional on $\mathbb{C}^n$ if T is a complex linear map from $\mathbb{C}^n$ to $\mathbb{C}$. Then it is an easy fact that for each $w \in \mathbb{C}^n$, the map

$$z \mapsto (z,w)$$

is a linear functional on $\mathbb{C}^n$. Now the claim is that these are all of them. In other words, all linear functionals on $\mathbb{C}^n$ arise this way. This is because if $(\mathbb{C}^n)^*$ denotes the space of all linear functionals of $\mathbb{C}^n$, then it is a complex vector space of dimension at most d (because any linear functional T on $\mathbb{C}^n$ is determined, by linearity, by their value on $T(e_j)$ where e_j is the unit vector in $\mathbb{C}^n$ with jth component $= 1$ and all other components being zero, and there are just at most d degrees of freedom). Now we have already exhibited a linear map from $\mathbb{C}^n$ into $(\mathbb{C}^n)^*$, namely

$$w \mapsto (z \mapsto (z,w)).$$

It is easy to see that this map is injective. Hence from dimension considerations, we see that this map is actually surjective.

Now suppose A is a linear map from the Hilbert space $\mathbb{C}^n$ to the Hilbert space $\mathbb{C}^m$, where n may be different from m. Then for each $w \in \mathbb{C}^m$, the map

$$z \mapsto (Az,w)$$

is a linear functional on $\mathbb{C}^n$. Hence by our discussion above, there is a vector $w' \in \mathbb{C}^n$ such that

$$(Az,w) = (z,w')$$

for all $z \in \mathbb{C}^n$. It is easy to see that for a given w there is just one such w', and we define this to be A^*w. This defines a complex linear map $A^*\colon \mathbb{C}^m \to \mathbb{C}^n$, and this is usually called the conjugate transpose of A. The same construction can be done whenever we have a bounded linear map

$$A\colon H_1 \to H_2$$

from a Hilbert space into another Hilbert space, and gives a bounded linear map

$$A^*\colon H_2 \to H_1.$$

This is usually called the adjoint of A. The crucial ingredient of proving the existence of A^* in this setting is the Riesz representation theorem, which says that every bounded linear functional on a Hilbert space H is of the form

$$z \mapsto (z,w)$$

for some $w \in H$. The proof of this is more difficult than its finite dimensional analogue, and you can find a proof of this in any functional analysis textbook.

Back to Radon transforms, there we are basically taking the usual L^2 inner product on $\mathcal{S}(\mathbb{R}^d)$, and the inner product on $\mathcal{S}(\mathbb{R} \times \mathbb{S}^{d-1})$ defined by

$$(F,G) = \int_{\mathbb{R}\times\mathbb{S}^{d-1}} F(t,\gamma)\overline{G(t,\gamma)}d\sigma(\gamma)dt.$$

Then the Radon transform

$$R\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R} \times \mathbb{S}^{d-1})$$

is a linear map, and its formal adjoint can be defined as above; note that since $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}\times\mathbb{S}^{d-1})$ are not complete as inner product spaces (hence not Hilbert spaces), the existence of the adjoint of R doesn't follow from the general considerations above. Nevertheless, one can show that the formal adjoint of the Radon transform exists, and maps

$$R^*\colon \mathcal{S}(\mathbb{R} \times \mathbb{S}^{d-1}) \to \mathcal{S}(\mathbb{R}^d).$$