MATH 152 FALL 2010 01-03/07-09
NOTES ON POWER SERIES AND TAYLOR POLYNOMIAL

This short note is intended to supplement Section 10.6-10.7 and 8.4 of Rogawski's Calculus. This is very brief, and does not cover all that is covered in class. It will also be much easier to read the notes if you attend the lectures.

A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

which defines a function of x if it converges. Usually we take $c = 0$.

There are usually four questions that one asks about power series:

(1) Suppose a power series is given (i.e. the a_n's and c are given). For which x does the series converge?
(2) Suppose a power series converges for a certain range of x, and therefore defines a function of x there. In this case we say the power series *represents* the function obtained. Is there a closed formula for the function that the power series represents?
(3) Suppose instead we are given a function to begin with. When can we find a convergent power series that represents this function?
(4) Suppose we are given a function, which we know *a priori* is representable by a convergent power series. How do we find the power series that represents the function?

In the first four sections that follows, we answer these questions one by one. In the last section we discuss Taylor polynomials, which allows us, among other things, to prove some formula that we will assume in the first four sections.

1. Defining a function by a power series

Suppose a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is given. It may or may not converge for any given x (except for $x = c$, where obviously it must converge). How do we know for which x it converges?

The answer is contained in the following theorem, and the key is the notion of the *radius of convergence* of a power series:

Theorem 1. *Suppose a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is given. Then there is a unique number $R \geq 0$ (which may be infinite) such that the series converges for all x that satisfies $|x-c| < R$ and diverges for all x that satisfies $|x-c| > R$.*

The number R in the above theorem is called the *radius of convergence* of the power series. Its value can usually be computed by the following theorem:

Theorem 2. *(a) If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ *exists, or is equal to* $+\infty$*, then* R *is equal to 1 over this limit.*
(b) If $\lim_{n\to\infty}|a_n|^{\frac{1}{n}}$ *exists, or is equal to* $+\infty$*, then* R *is equal to 1 over this limit.*

Here it is understood that $\frac{1}{0}=+\infty$, and $\frac{1}{+\infty}=0$.

For example, consider the power series $\sum_{n=0}^{\infty}\frac{(-2)^n}{(n+1)^2}(x-3)^n$. Since

$$\lim_{n\to\infty}\left|\frac{\frac{(-2)^{n+1}}{(n+2)^2}}{\frac{(-2)^n}{(n+1)^2}}\right| = \lim_{n\to\infty}\frac{2(n+1)^2}{(n+2)^2}=2,$$

the radius of convergence of this power series is $\frac{1}{2}$. Hence the power series converges for all x satisfying $|x-3|<\frac{1}{2}$, i.e. for all x in the interval $2.5<x<3.5$, and diverges for all x satisfying $|x-3|>\frac{1}{2}$, i.e. for all x satisfying $x>3.5$ or $x<2.5$.

Question: what happens, in the previous example, if $x=2.5$ or $x=3.5$?

Answer: There is no general answer to this question. The convergence at the end points $c-R$ and $c+R$ must be determined separately.

In this case the power series converges at both endpoints. When $x=2.5$, the power series is

$$\sum_{n=0}^{\infty}\frac{(-2)^n}{(n+1)^2}\left(-\frac{1}{2}\right)^n=\sum_{n=0}^{\infty}\frac{1}{(n+1)^2},$$

which is known to be convergent. When $x=3.5$, the power series is

$$\sum_{n=0}^{\infty}\frac{(-2)^n}{(n+1)^2}\left(\frac{1}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2},$$

which converges absolutely (hence converges).

See the book, Example 10.6.2, for an example where the power series fail to converge at one of the end points of the interval of convergence.

The above allows us to determine the range of x over which a given power series defines a function. For example, in the example above, the power series defines a function on the closed interval $2.5\le x\le 3.5$. This completely settles the first question in the introduction.

2. Finding a closed form for the function represented by a power series

In this section we move on to the second question. Suppose a function is defined by a given convergent power series for a certain range of x. Can we find a closed form for this function?

There are only a few instances where this is possible. In this course you are only required to know one class of examples; in fact we have covered this in a disguise already. This is the situation when the given power series is actually a geometric series. The most basic example is given by the power series

$$\sum_{n=0}^{\infty} x^n.$$

This is a geometric series. We know it converges for $|x| < 1$, and diverges for $|x| \geq 1$. (We know this even without having to compute the radius of convergence; this is just a result about geometric series!) Hence it defines a function on the open interval $-1 < x < 1$. What is the function that it defines there? In this case it is easy: our knowledge of geometric series tells us that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for all x with $|x| < 1$ (because the first term is 1 and the common ratio is x). This computes for us the function that the power series $\sum_{n=0}^{\infty} x^n$ represents already.

See also Examples 10.6.4 and 10.6.5 of the book for variants.

3. Expanding a given function in power series

Now we move on to the third question in the introduction. Suppose instead we are given a function of x. We ask whether we could find a convergent power series so that it is equal to the given function on a given range.

We have seen one instance when this is possible already. If the given function is $f(x) = \frac{1}{1-x}$, then we can certainly find a power series that is equal to this function on the open interval $-1 < x < 1$: It is the power series $1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$. It is easy to take the geometric power series and sum it to $\frac{1}{1-x}$; to go the other way round, and realize that the function $\frac{1}{1-x}$ can be expanded as a power series by using the geometric series formula, requires a bit of thought. This is one of the things that is slick enough, that you probably want to remember by heart. We record again this important formula:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad \text{if } |x| < 1.$$

It is important, however, to note that this formula fails to hold if $|x| > 1$, even though the function $f(x)$ is well-defined there.

The second example is when the given function is e^x, $\cos x$ or $\sin x$. In fact for any real number x,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

This formula is sometimes taken as the definition of e^x. You may continue to take this for granted in this course. You may also assume for granted the following power series expansion of $\cos x$ and $\sin x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

(We will provide, at the end, a proof of these three formula.)

The third example, which you can also take for granted, is the following formula for $(1+x)^a$ for any exponent a, if $|x|<1$:

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \ldots, \quad \text{if } |x| < 1.$$

(Again, the justification of this formula will come only at the end.)

Now these examples will allow us to expand, all of a sudden, a lot of functions in convergent power series.

Technique 1: Substitution in known power series

This is a trivial technique. One example is the following. Suppose we want to expand the function $\frac{1}{1-3x}$ in power series. Then remember the power series expansion of the function $\frac{1}{1-u}$:

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n = 1 + u + u^2 + u^3 + \ldots, \quad \text{for all } |u| < 1.$$

Let $u = 3x$. Then the assumption $|u| < 1$ is the same as the assumption $|x| < \frac{1}{3}$. Hence

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n = 1 + 3x + 9x^2 + 27x^3 + \ldots \quad \text{for all } |x| < \frac{1}{3}.$$

Another, even more basic example (which you may want to remember), is the power series expansion of $\frac{1}{1+x}$:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{if } |x| < 1.$$

This follows by substituting $u = -x$ in the power series expansion of $\frac{1}{1-u}$ (or can be derived directly using geometric series). Note the alternating signs in the series.

Compare with Examples 10.6.4, 10.6.5 and 10.7.6 of the book.

Technique 2: Adding or multiplying known power series

One example here is in the power series expansion of $x \sin x$. The power series expansion is just given by

$$x \sin x = x \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}.$$

See also Examples 10.7.4 and 10.7.5 of the book.

Technique 3: Differentiating or integrating a known power series.

This is based on the following theorem, which we state without proof:

Theorem 3. *Suppose $f(x)$ is a function that is represented by a convergent power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ in an open interval $c-R<x<c+R$. Then*

(a) $f(x)$ is differentiable on that open interval, and $f'(x)$ can be computed by differentiating term by term:

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}.$$

(The convergence of the power series on the right hand side of the above equation on the open interval is part of the assertion of the theorem.)

(b) If x lies in that open interval, then the definite integral of $f(x)$ from c to x can be computed by integrating term by term:

$$\int_c^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}.$$

(The convergence of the power series on the right hand side of the above equation on the open interval is again part of the assertion of the theorem.)

Below are some examples. These are important enough that you may want to remember by heart how to derive them.

In the first example, we expand $\ln(1-x)$ in a convergent power series: we claim

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{if } |x|<1.$$

To see this, the crucial observation here is that

$$\ln(1-x) = -\int_0^x \frac{1}{1-u} du \quad \text{if } |x|<1.$$

(Why is this true?) Since

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots$$

for $|u|<1$, integrating both sides of this equation from 0 to x, we get

$$\int_0^x \frac{1}{1-u} du = \int_0^x (1+u+u^2+u^3+\dots)du = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \dots$$

if $|x|<1$. (This is the assertion of the thoerem.) Our claim then follows.

Similarly, we expand $\ln(1+x)$ in a convergent power series:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad \text{if } |x|<1.$$

In the second example, we expand $\arctan x$ in a convergent power series: we claim

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad \text{if } |x|<1.$$

To see this, the key here is that

$$\arctan x = \int_0^x \frac{1}{1+u^2} du \quad \text{for all } x.$$

(You can check this formula using trigonometric substitution on the integral on the right hand side, or differentiate $\arctan x$ implicitly to see that its derivative is $\frac{1}{1+x^2}$). Now for $|u| < 1$,

$$\frac{1}{1+u^2} = 1 - u^2 + u^4 - u^6 + \dots = \sum_{n=0}^{\infty} (-1)^n u^{2n}.$$

Hence for $|x| < 1$, we have

$$\arctan x = \int_0^x (1 - u^2 + u^4 - u^6 + \dots) du = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

as desired.

In the third example, we expand $\frac{1}{(1-x)^2}$ in a convergent power series. There are two ways to do it: one is by multiplying the power series expansion of $\frac{1}{1-x}$ with itself, and another is to differentiate the power series expansion of $\frac{1}{1-x}$. The second method is simpler, and this is the one we present here.

The key observation here is that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right).$$

(Why is this true?) Now

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots = \sum_{n=0}^{\infty} x^n, \quad \text{if } |x| < 1.$$

By part (a) of the theorem, this says

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + \dots) = 1 + 2x + 3x^2 + 4x^3 + \dots$$

if $|x| < 1$. Hence

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{n=0}^{\infty} (n+1)x^n, \quad \text{if } |x| < 1.$$

Technique 4: Composing known power series

One example is the power series expansion of $\sec x = \frac{1}{\cos x}$. Since

$$\sec x = \frac{1}{\cos x} = \frac{1}{1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \dots\right)},$$

using the power series expansion $\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots$, the above is equal to (for small x)

$$\begin{aligned} & 1 + \left(\frac{x^2}{2} - \frac{x^4}{24} + \dots\right) + \left(\frac{x^2}{2} - \frac{x^4}{24} + \dots\right)^2 + \dots \\ =& 1 + \frac{x^2}{2} - \frac{x^4}{24} + \left(\frac{x^2}{2}\right)^2 + \text{ higher powers of } x \\ =& 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \text{ higher powers of } x \ . \end{aligned}$$

Compare with Example 10.7.5 of the book.

The above are some examples when one can actually expand a given function in power series. In general, however, there are functions that simply *cannot* be represented by a convergent power series at all. It is not that we do not have enough technology to do so; it is a fact of life (which can be proved rigorously) that there are some functions that are not limits of *any* convergent power series. In fact there are even such functions that are infinitely differentiable. In other words, there are funny infinitely differentiable functions, for which there is *no* power series at all that would converge to that function. To distinguish those nice functions that can be represented by a power series, we introduce the following definition:

Definition 1. *A function $f(x)$ is said to be real analytic on an interval $c - R < x < c + R$ if there is a convergent power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ such that*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

for all x in that interval.

We have seen that $\frac{1}{1-x}$ is real analytic on the interval $-1 < x < 1$, that e^x, $\cos x$ and $\sin x$ are real analytic on the whole real line, etc. In the next section, we assume that we are given a function $f(x)$ that is real analytic on the interval $c - R < x < c + R$, and try to find the coefficients a_n in the power series that is guaranteed to exist in the above definition. This answers our last question in the introduction.

4. Expanding a given function in power series, knowing that it is possible

Suppose we are given a function $f(x)$, which we *assume* can be represented by a convergent power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ on an interval $c - R < x < c + R$ for some $R > 0$. Suppose we want to find the coefficients a_n. It turns out this is really easy: we have the following theorem.

Theorem 4. *Suppose we* assume *that a function $f(x)$ can be represented by a convergent power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ on an interval $c - R < x < c + R$ for some $R > 0$. Then*

$$a_n = \frac{f^{(n)}(c)}{n!} \quad \text{for all } n \geq 0.$$

In other words, the coefficient a_n is simply the n-th derivative of f at the point c divided by $n!$. Since the function is given, the value of a_n can then be computed by differentiating the function.

The proof is very easy: remember we *assumed* that

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

for all $c - R < x < c + R$. By part (a) of Theorem 3, if one differentiates both sides term by term k times and evaluate at c, on the left hand side we have $f^{(k)}(c)$, and on the right hand side we get

$$a_k k! + a_{k+1}\frac{(k+1)!}{1!}(x-c) + a_{k+2}\frac{(k+2)!}{2!}(x-c)^2 + \dots\bigg|_{x=c} = a_k k!.$$

Hence $f^{(k)}(c) = a_k k!$ for all k, which is our claim.

Let's now look at some examples. In the first example, let's *assume* for the moment that e^x can be represented by a convergent power series $\sum_{n=0}^{\infty} a_n x^n$. Then the theorem says that

$$a_n = \frac{1}{n!} \frac{d^n}{dx^n} e^x \bigg|_{x=0} = \frac{1}{n!}$$

for all n, since any number of derivatives of e^x is always e^x, which evaluates to 1 at $x = 0$. This agrees with our previous formula for the power series expansion of e^x.

Similarly, *assume* for the moment that $\cos x$ can be represented by a convergent power series $\sum_{n=0}^{\infty} b_n x^n$. Then the theorem says that

$$b_n = \frac{1}{n!} \frac{d^n}{dx^n} (\cos x) \bigg|_{x=0}$$

for all n, which is

$$b_n = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ \frac{(-1)^{\frac{n}{2}}}{n!} & \text{if } n \text{ is even.} \end{cases}$$

This agrees with our previous formula for the power series expansion of $\cos x$. There is a similar story for $\sin x$ and $(1 + x)^a$.

These suggests that if e^x or $\cos x$ etc can be represented by a convergent power series, then the power series should be the ones given in the previous section. The problem here is that we did NOT know yet why e^x or $\cos x$ etc can be represented by a convergent power series. Thus the above discussion does NOT prove the power series expansion of e^x and $\cos x$. In fact, the problem with this theorem is that usually one does NOT know that a function can be represented by a convergent power series. One way of stating the current theorem is the following. Suppose we *assume* that a function $f(x)$ can be represented by a convergent power series $\sum_{n=0}^{\infty} a_n (x - c)^n$ on an interval $c - R < x < c + R$ for some $R > 0$. Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n \quad \text{for all } c - R < x < c + R.$$

Hence it may be tempted to suggest that if we are given an infinitely differentiable function $f(x)$, where the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$ converges for all $c-R < x < c+R$, then the power series must converge to $f(x)$, and $f(x)$ can be represented by a convergent power series on the interval $c-R < x < c+R$. This is NOT TRUE. There are pathological infinitely differentiable functions such that $f^{(n)}(0) = 0$ for all n, so that $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ converges and is zero for all x, and yet f is not identically zero. So Theorem 4 above is rarely useful in practice.

To be able to prove the formula for the power series expansions of e^x, $\cos x$ and $\sin x$, etc, we need to study Taylor polynomials. These are also of independent interest, and is covered in the next Section.

5. Taylor polynomials

Suppose we are given an infinitely differentiable function $f(x)$. The Nth Taylor polynomial of $f(x)$ around the point c is by definition

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

(We suppressed the dependence of f and c in the notation of $T_N(x)$ for brevity.) If we assume that $f(x)$ can be represented by a convergent power series on an interval $c - R < x < c + R$, then $T_N(x)$ converges to $f(x)$ on this interval as N tends to infinity. Hence we may suspect that the Nth Taylor polynomial of a function $f(x)$ is a good approximation of $f(x)$.

To make this precise, we have the following Theorems:

Theorem 5. *Suppose N is given, and $f(x)$ is infinitely differentiable*[1]*. Then*

$$f(x) - T_N(x) = \frac{1}{N!}\int_c^x (x-t)^N f^{(N+1)}(t)dt,$$

where $T_N(x)$ is the Taylor polynomial of f around the point c.

In fact the theorem follows from integration by parts (see p.506 of the book).

By estimating the integral in the above theorem, the following theorem follows:

Theorem 6. *Suppose N is given, and $f(x)$ is infinitely differentiable. Then*

$$|f(x) - T_N(x)| \le \frac{K_{N+1}}{(N+1)!}|x-c|^{N+1},$$

where $T_N(x)$ is the Taylor polynomial of f around the point c and K_{N+1} is the maximum of $|f^{(N+1)}(t)|$ for all t between c and x.

Thus if we fix N, $|f(x) - T_N(x)|$ gets smaller and smaller as x tends to c (or equivalently, $T_N(x)$ gets closer and closer to $f(x)$ as x tends to c).

For example, the Nth Taylor polynomial of e^x around 0 is

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!}.$$

(Note that this is a finite sum!) The theorem says that e^x is approximately $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!}$, in the sense that

$$\left| e^x - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!}\right)\right| \le \frac{K_{N+1}}{(N+1)!}|x|^{N+1}$$

for all x, where K_{N+1} is the maximum of the absolute value of the $(N+1)$th derivative of e^x on the interval between 0 and x. One can then use this formula to

[1] Technically, we only need f to have $N + 1$ continuous derivatives in this and the following theorem. But for simplicity, and this is what usually hold in applications anyway, we state the theorem for infinitely differentiable functions.

compute e^x to any accuracy: For example, to compute $e^{0.1}$ to two decimal places, one observes that for any N,

$$\left| e^{0.1} - \left(1 + 0.1 + \frac{0.1^2}{2!} + \cdots + \frac{0.1^N}{N!}\right) \right| \leq \frac{3}{(N+1)!} 0.1^{N+1},$$

because the absolute value of the $(N+1)$th derivative of e^x is still e^x, whose maximum over the interval $0 \leq x \leq 0.1$ is $e^{0.1}$, and $e^{0.1} \leq e \leq 3$. Now we choose N such that

$$\frac{3}{(N+1)!} 0.1^{N+1} < 0.005;$$

in fact a little trial and error shows that $N = 2$ will do. Therefore

$$\left| e^{0.1} - \left(1 + 0.1 + \frac{0.1^2}{2!}\right) \right| < 0.005,$$

i.e. $|e^{0.1} - 1.105| < 0.005$, and it follows that $e^{0.1} \simeq 1.11$ correct to 2 decimal places.

Section 8.4 of the book contains many more examples in detail.

Finally, let's prove the power series expansion of $\cos x$ using Theorem 6. In fact Theorem 6 asserts that

$$\left| \cos x - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^N x^{2N}}{(2N)!}\right) \right| \leq \frac{K_{2N+1}}{(2N+1)!} |x|^{2N+1}$$

for all N and all x, where K_{2N+1} is the maximum of the absolute value of the $(2N+1)$th derivative of $\cos x$ on the interval between 0 and x. Since any odd number of derivatives of $\cos x$ is either $\sin x$ or $-\sin x$, one sees that $K_{2N+1} \leq 1$ for any N. Hence we conclude that

$$\left| \cos x - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^N x^{2N}}{(2N)!}\right) \right| \leq \frac{1}{(2N+1)!} |x|^{2N+1}$$

for all N, and as N tends to infinity, the right hand side tends to 0 no matter what x is (c.f. Example 8 in Section 10.1 of the book). By the squeeze theorem, this shows that

$$\cos x = \lim_{N \to \infty} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^N x^{2N}}{(2N)!}\right),$$

i.e.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

This completes the proof of the power series expansion of $\cos x$. Similarly, one can prove the power series expansion of e^x, $\sin x$ for all real x, as well as that of $(1+x)^a$ for all $|x| < 1$, using a similar method.