MATH2242/MATH6242 Homework 3

Due Date: September 21, 2020.

1. Compute the coefficients of the first fundamental forms of the following surfaces:

   (a) $\mathbf{x}(u,v) = (a\sin u\cos v, b\sin u\sin v, c\cos u)$, $\quad (u,v) \in (0,\pi)\times(-\pi,\pi)$ $\quad$ (ellipsoid)

   (b) $\mathbf{x}(u,v) = (au\cosh v, bu\sinh v, u^2)$, $\quad (u,v) \in (0,\infty)\times\mathbb{R}$ $\quad$ (half of a hyperbolic paraboloid)

   (c) $\mathbf{x}(u,v) = (\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2})$ $\quad$ (unit sphere parametrized by stereographic projection)

   In other words, compute $E = \langle \mathbf{x}_u, \mathbf{x}_u\rangle$, $F = \langle \mathbf{x}_u, \mathbf{x}_v\rangle$, and $G = \langle \mathbf{x}_v, \mathbf{x}_v\rangle$. For each of the surfaces, it may be instructive to try to draw a picture, that features families of curves on which u is constant, and on which v is constant.

2. Show that the area A of a bounded region R of the surface $z = f(x,y)$ is

$$A = \iint_Q \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\, dxdy$$

   where Q is the projection of R onto the x-y plane. Use this to evaluate the area of the hemisphere $\{(x,y,z) \in \mathbb{R}^3 \colon x^2+y^2+z^2 = R^2, z > 0\}$ where $R > 0$ is the radius of the hemisphere.

3. Let $\mathbf{x}(u,v) = (\sin u\cos v, \sin u\sin v, \cos u)$ be a local parametrization of the unit sphere where $(u,v) \in (0,\pi)\times(-\pi,\pi)$ (this is the special case of Question 1(a) with $a = b = c = 1$). Let $\alpha \colon (-1,1) \to \mathbb{S}^2$ be a curve on the unit sphere defined by $\alpha(t) = \mathbf{x}(u(t), v(t))$ for some functions $u \colon (-1,1) \to (0,\pi)$ and $v \colon (-1,1) \to (-\pi,\pi)$. Find a formula for the length of α from $t = -1$ to $t = 1$. As a result, find the length of the latitude that corresponds to $\alpha(t) = \mathbf{x}(\frac{\pi}{2}+\theta, \pi t)$, $t \in (-1,1)$ where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is fixed once we fix the latitude.

4. Let $\mathbf{x}(u,v) = (u, v, u^2 - v^2)$ for $(u,v) \in \mathbb{R}^2$; it parametrizes the hyperbolic paraboloid $z = x^2 - y^2$. Let's orient this hyperbolic paraboloid with a unit normal N that has a positive third component in $\mathbb{R}^3$.

   (a) Find an explicit expression for this unit normal N at $\mathbf{x}(u,v)$ for every $(u,v) \in \mathbb{R}^2$.

   (b) Hence, evaluate the coefficients of the second fundamental form, namely

$$e = \langle \mathbf{x}_{uu}, N\rangle, \quad f = \langle \mathbf{x}_{uv}, N\rangle, \quad g = \langle \mathbf{x}_{vv}, N\rangle.$$

   (c) In addition, directly from the expression you obtained for N, compute

$$dN_p(\mathbf{x}_u), \quad dN_p(\mathbf{x}_v) \quad \text{where we abbreviated } p = \mathbf{x}(u,v).$$

   (Hint: Differentiate $N(\mathbf{x}(u,v))$ with respect to u and v respectively.)

   (d) Finally, use this to verify that indeed

$$e = -\langle \mathbf{x}_u, dN_p(\mathbf{x}_u)\rangle, \quad f = -\langle \mathbf{x}_u, dN_p(\mathbf{x}_v)\rangle = -\langle \mathbf{x}_v, dN_p(\mathbf{x}_u)\rangle, \quad g = -\langle \mathbf{x}_v, dN_p(\mathbf{x}_v)\rangle$$

   in this specific example, without resorting to general theory.