MATH2242/MATH6242 Homework 4

Due Date: October 6, 2020.

1. Fix $\phi \in (0, \frac{\pi}{2})$. Let
$$\mathbf{x}(u,v) = (u\cos v \sin\phi, u \sin v \sin\phi, u\cos\phi), \quad (u,v) \in (0,\infty)\times(-\pi,\pi)$$
be a parametrization of a cone $z = \frac{\sqrt{x^2+y^2}}{\tan\phi}$ (of aperture ϕ) minus the line through the origin in the direction $(-1, 0, \cot\phi)$.

   (a) Sketch this cone S. Indicate what it means for this cone to have aperture ϕ. Draw in red a few curves where u is constant, and draw in blue a few curves where v is constant.

   (b) Find the inward unit normal N to this cone at $\mathbf{x}(u,v)$. Henceforth we orient this cone by this inward unit normal.

   (c) Compute the coefficients of the second fundamental form in this parametrization, i.e. $e = \langle \mathbf{x}_{uu}, N\rangle$, $f = \langle \mathbf{x}_{uv}, N\rangle$, $g = \langle \mathbf{x}_{vv}, N\rangle$, and evaluate these at the point $(\sin\phi, 0, \cos\phi)$.

   (d) The second fundamental form on this cone is defined to be
$$\mathbf{II}_p(w) = -\langle dN_p(w), w\rangle \quad \text{for } w \in T_p(S).$$
Using part (c), find $\mathbf{II}_p(8\mathbf{x}_u + 9\mathbf{x}_v)$ if $p = \mathbf{x}(6,4)$.

   (e) (i) By differentiating the identity $\langle N, N\rangle = 1$, show that $dN(\mathbf{x}_u)$ and $dN(\mathbf{x}_v)$ are both orthogonal to N at $\mathbf{x}(u,v)$.

   (ii) If
$$\begin{cases} dN(\mathbf{x}_u) = a_{11}\mathbf{x}_u + a_{12}\mathbf{x}_v \\ dN(\mathbf{x}_v) = a_{21}\mathbf{x}_u + a_{22}\mathbf{x}_v \end{cases},$$
find a_{11}, a_{12}, a_{21} and a_{22}. Using this, find the Gaussian and the mean curvatures of this cone at a general point $\mathbf{x}(u,v)$.

   (f) Let $\gamma(t)$ be a curve on this cone satisfying $\gamma(0) = (1, 0, \cot\phi)$ and $\gamma'(0) = (\sin\phi, -1, \cos\phi)$. Find the normal curvature of $\gamma(t)$ at $t = 0$.

2. Fix two positive real numbers a and b. Let
$$\mathbf{x}(u,v) = (a\cos u\cos v, a\cos u \sin v, b\sin u), \quad (u,v) \in (-\frac{\pi}{2}, \frac{\pi}{2})\times(-\pi,\pi)$$
be a parametrization of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$ minus a longitude, and orient the ellipsoid by choosing the outward unit normal vector N.

   (a) Compute the coefficients of the second fundamental form in this parametrization, i.e. $e = \langle \mathbf{x}_{uu}, N\rangle$, $f = \langle \mathbf{x}_{uv}, N\rangle$, $g = \langle \mathbf{x}_{vv}, N\rangle$, and evaluate these at the point $(\frac{a}{\sqrt{2}}, 0, \frac{b}{\sqrt{2}})$.

   (b) Find the Gaussian curvature and the mean curvature of this ellipsoid at a general point $\mathbf{x}(u,v)$. Hence evaluate the Gaussian and the mean curvatures at the point $(a, 0, 0)$.

   (c) Let N_u be a shorthand for $dN_p(\mathbf{x}_u)$ where $p = \mathbf{x}(u,v)$. If $N_u = A(u,v)\mathbf{x}_u(u,v) + B(u,v)\mathbf{x}_v(u,v)$, find $A(u,v)$ and $B(u,v)$. In particular, express N_u in terms of $\mathbf{x}_u$ and $\mathbf{x}_v$ at the point $(0, \frac{a}{2}, \frac{\sqrt{3}}{2}b)$.

   (d) Find the principal curvatures of this ellipsoid at a general point $\mathbf{x}(u,v)$. What are the principal directions at the point $(\frac{a}{2}, \frac{a}{2}, \frac{b}{\sqrt{2}})$?

   (e) Show that $\mathbf{x}_u$ and $\mathbf{x}_v$ are principal directions to this ellipsoid at any $(u,v) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\pi, \pi)$. Are they orthogonal to each other?

3. Let S be a regular surface in $\mathbb{R}^3$ and p_1, p_2, p_3 be three points on S.

   (a) If the principal curvatures of S at p_1 are 5 and -7, find the Gaussian curvature and the mean curvature of S at p_1.

   (b) If the Gaussian curvature and the mean curvature of S at p_2 are -24 and 1 respectively, find the principal curvatures of S at p_2.

   (c) Suppose there exists two non-zero tangent vectors $v_1, v_2 \in T_{p_3}(S)$ such that

   $$dN_{p_3}(v_1) = 8v_1 \quad \text{and} \quad dN_{p_3}(v_2) = (0, 0, 0).$$

   Can you determine the Gaussian and the mean curvatures of S at p_3? Can you say something about the angle between v_1 and v_2 at p_3? Explain your answers.

4. Let S be a regular surface in $\mathbb{R}^3$ and $\alpha\colon I \to \mathbb{R}^3$ be a regular curve in $\mathbb{R}^3$ lying entirely on S. Show that for any $t_0 \in I$, the normal curvature k_n of $\alpha(t)$ at $t = t_0$ is always less than or equal to the curvature of $\alpha(t)$ at $t = t_0$. (Hint: Start from the definition of the normal curvature and take absolute value. Then apply Cauchy-Schwarz. The proof is just one line.)

5. Let

   $$\mathbf{x}(u, v) = (u, v, \frac{u^2 + v^2}{2}), \quad (u, v) \in \mathbb{R}^2$$

   be a parametrization of the paraboloid $z = \frac{x^2+y^2}{2}$. Compute the dot products of $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, $\mathbf{x}_{vv}$ with $\mathbf{x}_u$ and $\mathbf{x}_v$ at a general point $\mathbf{x}(u, v)$. Use these to compute the Christoffel symbols Γ_{ij}^k, for $i, j, k \in \{1, 2\}$. Hence evaluate the covariant derivatives $\nabla_{\mathbf{x}_u}\mathbf{x}_u$, $\nabla_{\mathbf{x}_v}\mathbf{x}_u$ and $\nabla_{\mathbf{x}_v}\mathbf{x}_v$ at the point $p = (1, 1, 1)$ on this paraboloid.