MATH2242/MATH6242 Homework 5

Due Date: October 20, 2020.

1. Let $\mathbf{x}\colon U \to \mathbf{x}(U) \subset \mathbb{R}^3$ be a local parametrization of a regular surface S in $\mathbb{R}^3$, and E, F, G be the coefficients of the first fundamental form in the parametrization $\mathbf{x}$.

   (a) Show that
   $$\mathbf{x}_{uv} \cdot \mathbf{x}_u = \frac{1}{2}E_v \quad \text{and} \quad \mathbf{x}_{vv} \cdot \mathbf{x}_u = F_v - \frac{1}{2}G_u$$
   where the subscripts u and v denote partial derivatives with respect to u and v.

   (b) If $E = 1$, $F = 0$ and $G = 1 + u^2$ at every point of U (this is the case e.g. if we parametrize a helicoid by $\mathbf{x}(u, v) = (u \cos v, u \sin v, v))$, find Γ^1_{12} and Γ^2_{12} at the point $(u, v) = (2, 3)$. Hence express $\nabla_{\mathbf{v}}\mathbf{w}$ at $(u, v) = (2, 3)$ as a linear combination of $\mathbf{x}_u$ and $\mathbf{x}_v$ if $\mathbf{w} = (\cos u + uv^3)\mathbf{x}_u$ and $\mathbf{v} = (1 + u^2)\mathbf{x}_v$.

2. Let $\mathbf{x}\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\pi, \pi) \to S$ be a local parametrization of an ellipsoid S, and E, F, G be the coefficients of the first fundamental form in this local coordinate chart. Suppose
   $$E = a^2 \sin^2 u + c^2 \cos^2 u, \quad F = 0, \quad G = a^2 \cos^2 u$$
   where $a, c > 0$ are constants (this is the case, e.g. if $\mathbf{x}(u, v) = (a \cos u \cos v, a \cos u \sin v, c \sin u)$ as in Assignment 3).

   (a) Express $\nabla_{\mathbf{x}_u}\mathbf{x}_u$ and $\nabla_{\mathbf{x}_v}\mathbf{x}_u$ as a linear combination of $\mathbf{x}_u$ and $\mathbf{x}_v$.

   (b) Show that
   $$\nabla_{\mathbf{x}_v}\nabla_{\mathbf{x}_u}\mathbf{x}_u = -\frac{(a^2 - c^2)\sin^2 u}{a^2 \sin^2 u + c^2 \cos^2 u}\mathbf{x}_v, \quad \text{and} \quad \nabla_{\mathbf{x}_u}\nabla_{\mathbf{x}_v}\mathbf{x}_u = -\mathbf{x}_v.$$
   Hence show that
   $$R^2_{211} = \frac{c^2}{a^2 \sin^2 u + c^2 \cos^2 u} \quad \text{and} \quad R_{2112} = \frac{a^2 c^2 \cos^2 u}{a^2 \sin^2 u + c^2 \cos^2 u}.$$
   Deduce from this that the Gaussian curvature of the ellipsoid S at $\mathbf{x}(u, v)$ is
   $$K = \frac{c^2}{(a^2 \sin^2 u + c^2 \cos^2 u)^2}$$
   (which agrees with the answer we found in Assignment 4).

3. (a) Let $\mathbf{x}\colon U \to \mathbf{x}(U) \subset \mathbb{R}^3$ be a local parametrization of a regular surface S in $\mathbb{R}^3$, and E, F, G be the coefficients of the first fundamental form in the parametrization $\mathbf{x}$. Suppose E, F, G are all constant functions on U. What can you say about the Gaussian curvature of S on $\mathbf{x}(U)$? (Hint: What can you say about the Christoffel symbols first?)

   (b) Let $\mathbf{x}\colon U \to \mathbf{x}(U) \subset \mathbb{R}^3$ be a local parametrization of a regular surface S in $\mathbb{R}^3$. Show that the coefficients of the first and second fundamental forms in the parametrization $\mathbf{x}$ cannot satisfy all of the following at once:
   $$E = 1, \quad F = 0, \quad G = 1 \quad \text{and} \quad e = 1, \quad f = 0, \quad g = 1.$$
   (Hint: Compare the curvature tensor R_{2112} with the determinant of the second fundamental form, namely $eg - f^2$.)

4. Let $\mathbf{x}\colon \mathbb{R}^2 \to S$ be a local parametrization of a regular surface S in $\mathbb{R}^3$, and E, F, G be the coefficients of the first fundamental form in the parametrization $\mathbf{x}$. Suppose F is identically 0.

   (a) Show that the Gaussian curvature at the point $\mathbf{x}(u, v)$ is given by

   $$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v\right]$$

   where the subscripts u and v denote partial differentiation.

   (b) If we further have $E = G = \lambda$ for some function λ of $(u, v) \in \mathbb{R}^2$, then the Gaussian curvature at the point $\mathbf{x}(u, v)$ is given by

   $$K = -\frac{1}{2\lambda}\Delta(\log \lambda)$$

   where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the Laplacian on the (u, v) plane.

   (c) Hence, find the Gaussian curvature of S at $\mathbf{x}(1, 0)$ if $E = G = \cosh^2 u$ and $F = 0$ under some local parametrization $\mathbf{x}$ (this will be the case e.g. if

   $$\mathbf{x}(u, v) = (\cosh u \cos v, \cosh u \sin v, u)$$

   parametrizes the catenoid in $\mathbb{R}^3$).

5. Let $\mathbf{x}(u, v) = (\cos u \cos v, \cos u \sin v, 4 \sin u)$ be a local parametrization of an ellipsoid S, where $(u, v) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\pi, \pi)$. Let $\gamma(t) = \mathbf{x}(\frac{\pi}{6}, 7t)$ where $t \in (-\frac{\pi}{7}, \frac{\pi}{7})$.

   (a) Show that along γ, the Christoffel symbols $\Gamma^1_{12}, \Gamma^2_{12}, \Gamma^1_{22}$ and Γ^2_{22} are constants; furthermore, we have

   $$\Gamma^1_{12} = 0, \quad \Gamma^2_{12} = -\frac{1}{\sqrt{3}}, \quad \Gamma^1_{22} = \frac{\sqrt{3}}{49}, \quad \Gamma^2_{22} = 0.$$

   (b) Let $\mathbf{w}(t) = w_1(t)\mathbf{x}_u + w_2(t)\mathbf{x}_v$ be a vector field along the curve γ. Show that

   $$\frac{D\mathbf{w}}{dt} = \left(w_1'(t) + \frac{\sqrt{3}}{7}w_2(t)\right)\mathbf{x}_u + \left(w_2'(t) - \frac{7}{\sqrt{3}}w_1(t)\right)\mathbf{x}_v.$$

   (c) Suppose now $\mathbf{w}(t)$ is the parallel transport of $\gamma'(0)$ along γ. By solving a system of ordinary differential equations, show that

   $$\mathbf{w}(t) = -\sqrt{3}\sin(t)\mathbf{x}_u + 7\cos(t)\mathbf{x}_v$$

   for $t \in (-\frac{\pi}{7}, \frac{\pi}{7})$.