MATH2242/MATH6242 Homework 6

No need to turn in.

1. Let $R > 0$, and let $\mathbf{x}\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\pi, \pi) \to S$ be a local parametrization of the sphere S of radius 2 in $\mathbb{R}^3$, given by

$$\mathbf{x}(u,v) = (2\cos u \cos v, 2\cos u \sin v, 2\sin u), \quad (u,v) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\pi, \pi).$$

Let

$$\Omega = \Big\{\mathbf{x}(u,v) \in S\colon u \in [0, \frac{\pi}{3}], v \in [0, \frac{\pi}{2}]\Big\},$$

and let $\partial\Omega$ be parametrized by the union of 4 curves, given by

$$\gamma_1 = \mathbf{x}(0,t), \qquad t \in [0, \frac{\pi}{2}],$$

$$\gamma_2 = \mathbf{x}\Big(t, \frac{\pi}{2}\Big), \qquad t \in [0, \frac{\pi}{3}],$$

$$\gamma_3 = \mathbf{x}\Big(\frac{\pi}{3}, \frac{\pi}{2} - t\Big), \qquad t \in [0, \frac{\pi}{2}],$$

$$\gamma_4 = \mathbf{x}\Big(\frac{\pi}{3} - t, 0\Big), \qquad t \in [0, \frac{\pi}{3}].$$

We will orient S by the outward unit normal, i.e. we will take

$$N(\mathbf{x}(u,v)) = (\cos u \cos v, \cos u \sin v, \sin u).$$

(a) Draw a picture of the above configuration.

(b) Suppose one travels along $\partial\Omega$ by following consecutively $\gamma_1, \gamma_2, \gamma_3$ and then γ_4. Find the angle turned at each point where $\partial\Omega$ is not smooth.

(c) Compute the covariant derivative $\frac{D}{dt}\left(\frac{d\gamma_1}{dt}\right)$. Hence show that γ_1 is a geodesic on S. Similarly, show that γ_2 and γ_4 are both geodesics on S.

(d) Compute the geodesic curvature k_g along γ_3.

(e) Find the Gaussian curvature of the sphere at every point $\mathbf{x}(u,v) \in \Omega$.

(f) Using the local Gauss-Bonnet formula, find the area of Ω.

(g) The area of Ω can also be calculated by evaluating the integral

$$\int_0^{\frac{\pi}{3}} \int_0^{\frac{\pi}{2}} |\mathbf{x}_u \wedge \mathbf{x}_v| dv du.$$

Does your answer in part (f) agree with the value of this integral?

2. Let S be the surface of revolution obtained by rotating a unit circle $(x-2)^2 + y^2 = 1$ centered at $(2,0)$ around the z axis; it is then a torus, and

$$\mathbf{x}(u,v) = ((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u), \qquad (u,v) \in (-\pi,\pi) \times (-\pi,\pi)$$

parametrizes this torus except that it'll miss two circles on the torus.

(a) Using the global Gauss-Bonnet theorem, evaluate the total integral of the Gaussian curvature on S, i.e. evaluate

$$\iint_S K dA.$$

(You do not need to provide a justification for the Euler characteristic of the torus.)

(b) Show that the Gaussian curvature of S at $\mathbf{x}(u, v)$ is given by

$$K(\mathbf{x}(u, v)) = \frac{\cos u}{2 + \cos u}.$$

Also show that $dA = (2 + \cos u)dudv$. Hence, compute the integral in part (a) directly, without using the global Gauss-Bonnet theorem. Do your answers agree? Which method is easier?