Plane curves

Two main ways to describe a curve in the plane $\mathbb{R}^2$:

① Zero set of a (nice) function on $\mathbb{R}^2$

eg. $F(x,y) = x^2+y^2-1$

Then the set of points where $F(x,y)=0$ forms a circle.

② Image of a (nice) map from $\mathbb{R}$ to $\mathbb{R}^2$

eg. $\gamma(t) = (\cos t, \sin t)$

Then the image of $[0,2\pi]$ under γ parametrizes the same circle as above.

We will mainly use the second way :

we study parametrized curves in $\mathbb{R}^2$ (later in $\mathbb{R}^3$).

Definition. A (smooth) parametrized curve in $\mathbb{R}^2$ is a C^∞ map $\alpha: I \to \mathbb{R}^2$ of an open interval $I \subseteq \mathbb{R}$ into $\mathbb{R}^2$

This means $\alpha(t) = (x(t), y(t))$ where $x(t), y(t)$ are C^∞ functions of $t \in I$ (all curves assumed smooth unless stated otherwise).

Think of a parametrized curve as the trajectory of a particle. In particular, two different parametrized curves may trace out the same image:

eg. $\alpha(t) = (\cos t, \sin t)$ and $\beta(t) = (\cos(2t), \sin(2t))$ are considered different as parametrized curves.

If $\alpha(t)$ is a parametrized curve, then $\alpha(-t)$ traces out the same image in the opposite orientation.

In geometry, the ultimate interest are usually in (geometric) quantities that are defined independent of parametrizations.

Let $\alpha(t) = (x(t), y(t))$, $t \in I$ be a parametrized curve.

Definition. The tangent vector to α at t is

$$\alpha'(t) := (x'(t), y'(t))$$

The speed of α at t is $|\alpha'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$.

Definition. The length of α on an interval $[a,b] \subseteq I$ is

$$\int_a^b |\alpha'(t)|\,dt.$$

α is said to be parametrized by arc length, if

$$|\alpha'(t)| = 1 \quad \forall t \in I.$$

Exercise. Arc length of the parabola $\alpha(t) = (t, t^2)$ on $[0,1]$?

Answer: $\dfrac{2\sqrt{5} + \log(2+\sqrt{5})}{4}$.

Definition. A parametrized curve α is said to be regular, if
$$|\alpha'(t)| \neq 0 \quad \forall t \in I$$
(All curves assumed regular unless stated otherwise).

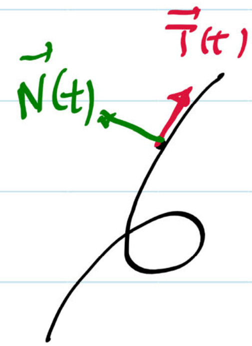
Theorem. Every regular parametrized curve may be reparametrized by arc length. More precisely, if $\alpha(t)$ is a regular parametrized curve on I, then $\exists$ C^∞ strictly increasing bijection $t: \tilde{I} \to I$ on some interval $\tilde{I}$, so that $\tilde{\alpha}(s) := \alpha(t(s))$ is parametrized by arc length on $\tilde{I}$.

In fact, if $I = (t_0, t_1)$, then $\tilde{I} = (0, L)$ where $L := \int_{t_0}^{t_1} |\alpha'(t)|\, dt$ and $t: \tilde{I} \to I$ is the inverse function to $s: I \to \tilde{I}$ defined by
$$s(t) := \int_{t_0}^{t} |\alpha'(u)|\, du \quad \text{for } t \in I.$$
Then $\alpha(I) = \tilde{\alpha}(\tilde{I})$, and $\tilde{\alpha}'(s) = \dfrac{\alpha'(t)}{|\alpha'(t)|}$ if $t = t(s)$ which shows $|\tilde{\alpha}'(s)| = 1$.

Let α be a regular curve defined on an interval I (not necessarily parametrized by arc length).

Definition. The unit tangent vector to α at t is

$$\vec{T}(t) := \frac{\alpha'(t)}{|\alpha'(t)|}.$$



The unit normal vector to α at t is the rotation of $\vec{T}(t)$ counterclockwise by $\frac{\pi}{2}$, denoted $\vec{N}(t)$. (If $\vec{T}(t) = (T_1(t), T_2(t))$ then $\vec{N}(t) = (-T_2(t), T_1(t))$.)

Since $\vec{T}(t) \cdot \vec{T}(t) = 1$ on I, differentiating with respect to t yields

$$\vec{T}'(t) \cdot \vec{T}(t) = 0$$

so $\vec{T}'(t)$ is always orthogonal to $\vec{T}(t) \Rightarrow \vec{T}'(t)$ is always a multiple of $\vec{N}(t)$

[Product rule for dot product of vector-valued functions: $(\alpha \cdot \beta)' = \alpha' \cdot \beta + \alpha \cdot \beta'$]

$|\vec{T}'(t)|$ measures how fast the unit tangent vector is changing, if we normalize by the speed of the curve.

Definition. The curvature $k(t)$ of a regular curve α is the unique real number such that $\dfrac{\vec{T}'(t)}{|\alpha'(t)|} = k(t)\vec{N}(t)$.

The above definitions are invariant under reparametrization of α. The above quantities are most easily calculated if α is parametrized by arc length so that $|\alpha'(s)| = 1$. In that case,

$$\vec{T}(s) = \alpha'(s), \quad \vec{T}'(s) = \alpha''(s) = k(s)\vec{N}(s) \quad \text{so} \quad \vec{N}(s) = \pm\frac{\alpha''(s)}{|\alpha''(s)|}.$$

This shows that $\vec{T}'(s) = \alpha''(s)$ is the acceleration vector to α, and $\vec{N}(s) = \pm\dfrac{\alpha''(s)}{|\alpha''(s)|}$ is the normalized acceleration vector.

We also have

$$|k(s)| = \text{length of the acceleration vector } \alpha''(s).$$

Exercise. Calculate $\vec{T}, \vec{N}, k$ for the curve

$$\alpha(s) = \left(6\cos\frac{s}{6},\ 6\sin\frac{s}{6}\right)$$

Answer: $\vec{T}(s) = \left(-\sin\frac{s}{6},\ \cos\frac{s}{6}\right)$

$\vec{N}(s) = \left(-\cos\frac{s}{6},\ -\sin\frac{s}{6}\right)$

$k(s) = \frac{1}{6}$.

Exercise. Calculate $\vec{T}, \vec{N}, k$ for the curve

$$\alpha(t) = (6\cos 4t,\ 6\sin 4t).$$

Answer: $\vec{T}(t) = (-\sin 4t,\ \cos 4t)$

$\vec{N}(t) = (-\cos 4t, -\sin 4t)$

$k(t) = \frac{1}{6}$.

Consider now a (regular) curve $\alpha(s)$ parametrized by arc length. We saw that $\vec{T}'(s) = k(s)\vec{N}(s)$, and we claim now $\vec{N}'(s) = -k(s)\vec{T}(s)$.
In fact, $\{\vec{T}(s), \vec{N}(s)\}$ form an orthonormal frame along α (called Frenet frame), and differentiating

$$\vec{N}(s)\cdot\vec{N}(s) = 1 \quad , \quad \vec{N}(s)\cdot\vec{T}(s) = 0$$

we obtain

$$\vec{N}'(s)\cdot\vec{N}(s) = 0, \quad \vec{N}'(s)\cdot\vec{T}(s) = -\vec{N}(s)\cdot\vec{T}'(s) = -k(s),$$

verifying our claim. Thus we have Frenet's formula (valid for arc length parametrized curves α):

$$\begin{cases} \vec{T}'(s) = k(s)\vec{N}(s) \\ \vec{N}'(s) = -k(s)\vec{T}(s) \end{cases} \qquad \left(\text{matrix form: } \begin{pmatrix} \vec{T}' \\ \vec{N}' \end{pmatrix} = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}\begin{pmatrix} \vec{T} \\ \vec{N} \end{pmatrix}\right)$$

Chain rule then shows that Frenet's formula must be modified, when $\alpha(t)$ is not necessarily parametrized by arc length:

$$\begin{cases} \vec{T}'(t) = |\alpha'(t)|\,k(t)\vec{N}(t) \\ \vec{N}'(t) = -|\alpha'(t)|\,k(t)\vec{T}(t) \end{cases} \qquad \left(\text{matrix form: } \begin{pmatrix} \vec{T}' \\ \vec{N}' \end{pmatrix} = |\alpha'|\begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}\begin{pmatrix} \vec{T} \\ \vec{N} \end{pmatrix}\right)!$$

Let $\alpha(s)$ be parametrized by arc length.

Frenet's formula allows us to Taylor expand $\alpha(s)$ around any s_0.

For simplicity, take $s_0 = 0$. Then we claim

$$\alpha(s) = \alpha(0) + \left(s - \frac{k^2(0)}{6} s^3\right) \vec{T}(0) + \left(\frac{k(0)}{2} s^2 + \frac{k'(0)}{6} s^3\right) \vec{N}(0) + O(s^4).$$

In fact, if $f(s) = [\alpha(s) - \alpha(0)] \cdot \vec{T}(0)$, then

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -k^2(0);$$

eg. by Frenet's formula,

$$f'''(0) = \alpha'''(0) \cdot \vec{T}(0) = (k\vec{N})'(0) \cdot \vec{T}(0) = k(0)\, \vec{N}'(0) \cdot \vec{T}(0) = -k^2(0).$$

Similarly, if $g(s) = [\alpha(s) - \alpha(0)] \cdot \vec{N}(0)$, then

$$g(0) = 0, \quad g'(0) = 0, \quad g''(0) = k(0), \quad g'''(0) = k'(0).$$

In particular, the projection of $\alpha(s) - \alpha(0)$ to $\vec{N}(0)$ vanishes to second order in s.

Space curves

Definition. A (smooth) parametrized curve in $\mathbb{R}^3$ is a C^∞ map $\alpha: I \to \mathbb{R}^3$ of an open interval $I \subseteq \mathbb{R}$ into $\mathbb{R}^3$.
It is said to be regular, if $|\alpha'| \neq 0$ on I.
It is said to be parametrized by arc length, if $|\alpha'| = 1$ on I.

Every regular curve in $\mathbb{R}^3$ can be reparametrized by arc length.

Let α be a regular curve, not necessarily parametrized by arc length.

Definition. The unit tangent vector to α at t is then $\vec{T}(t) = \dfrac{\alpha'(t)}{|\alpha'(t)|}$.

Exercise. Show that $\vec{T}'(t)$ is still always orthogonal to $\vec{T}(t)$

Definition. If $|\vec{T}'(t)| \neq 0$, the unit normal vector to α at t is then $\vec{N}(t) := \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$, and the unit binormal vector to α at t is then $\vec{B}(t) = \vec{T}(t) \wedge \vec{N}(t)$ (cross product in $\mathbb{R}^3$).

$\{\vec{T}, \vec{N}, \vec{B}\}$ form an orthonormal frame along α, called Frenet frame.

Exercise. Find the Frenet frame along $\alpha(s) = (\cos s, \frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\sin s)$.

Answer: $\vec{T}(s) = (-\sin s, \frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}\cos s)$

$\vec{N}(s) = (-\cos s, -\frac{1}{\sqrt{2}}\sin s, -\frac{1}{\sqrt{2}}\sin s)$

$\vec{B}(s) = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Two other ways to calculate the Frenet frame:

① Apply Gram-Schmidt process to the ordered basis $\{\alpha'(t), \alpha''(t), \alpha'''(t)\}$

The resulting ordered orthonormal frame is then $\{\vec{T}(t), \vec{N}(t), \vec{B}(t)\}$.

② Use the formula $\vec{T}(t) = \frac{\alpha'(t)}{|\alpha'(t)|}$, $\vec{B}(t) = \frac{\alpha'(t) \wedge \alpha''(t)}{|\alpha'(t) \wedge \alpha''(t)|}$, $\vec{N}(t) = \vec{B}(t) \wedge \vec{T}(t)$.

Both method works because $\text{span}\{\alpha'\} = \text{span}\{\vec{T}\}$ and $\text{span}\{\alpha', \alpha''\} = \text{span}\{\vec{T}, \vec{N}\}$.

Exercise. Find the Frenet frame along $\alpha(t) = (3t, 4t, 5t^2)$

Answer: $\vec{T}(t) = \frac{(3, 4, 10t)}{5\sqrt{1+4t^2}}$, $\vec{N}(t) = \frac{(-6t, -8t, 5)}{5\sqrt{1+4t^2}}$, $\vec{B}(t) = \frac{(4, -3, 0)}{5}$.

Consider now a curve $\alpha(s)$ in $\mathbb{R}^3$ parametrized by arc length. We will derive Frenet's formula and use it to define the curvature and the torsion of α.

transpose

Let $A(s)$ be a 3×3 matrix $A(s) = \begin{pmatrix} \vec{T}(s) \\ \vec{N}(s) \\ \vec{B}(s) \end{pmatrix}$ so that $A(s)A^t(s) = I$

Differentiating with respect to s:

$$\frac{dA}{ds}A^t(s) + A(s)\frac{dA^t}{ds} = 0 \implies \frac{dA}{ds}A^t(s) = -\left(\frac{dA}{ds}A^t(s)\right)^t$$

so $\frac{dA}{ds}A^t(s)$ is skew-symmetric for all s. Thus

$$\frac{dA}{ds}A^t(s) = \begin{pmatrix} 0 & k(s) & \mu(s) \\ -k(s) & 0 & -\tau(s) \\ -\mu(s) & -\tau(s) & 0 \end{pmatrix} \text{ for some functions } k, \tau, \mu.$$

But $\mu(s) = \vec{T}'(s)\cdot\vec{B}(s) = |\vec{T}'(s)|\,\vec{N}(s)\cdot\vec{B}(s) = 0$. Multiplying both sides by $A(s)$:

$$\frac{dA}{ds} = \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} A(s), \text{ i.e. } \begin{pmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{pmatrix} = \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix}\begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix}$$

which is Frenet's formula, valid for arc length parametrized curves for which $\vec{T}' \neq 0$. $k(s)$ and $\tau(s)$ are called the curvature and torsion of α resp.

$k(s)$ and $\tau(s)$ satisfy $\begin{cases} k(s) = |\vec{T}'(s)| = |\alpha''(s)| = -\vec{N}'(s)\cdot\vec{T}(s) \\ \tau(s) = \vec{B}'(s)\cdot\vec{N}(s) = -\vec{N}'(s)\cdot\vec{B}(s) \end{cases}$

The curvature $k(s)$ measures how fast $\vec{T}(s)$ changes; the torsion $\tau(s)$ measures how fast $\vec{N}(s)$ changes in the direction of $\vec{B}(s)$.

The Frenet formula can be more explicitly written as

$$\begin{cases} \vec{T}'(s) = k(s)\vec{N}(s) \\ \vec{N}'(s) = -k(s)\vec{T}(s) - \tau(s)\vec{B}(s) \\ \vec{B}'(s) = \tau(s)\vec{N}(s) \end{cases}$$ for arc length parametrized curves.

They can be used to Taylor expand such $\alpha(s)$ as

$$\alpha(s) = \alpha(0) + \left(s - \frac{k^2(0)}{6}s^3\right)\vec{T}(0) + \left(\frac{k(0)}{2}s^2 + \frac{k'(0)}{6}s^3\right)\vec{N}(0) - \frac{k(0)\tau(0)}{6}s^3\vec{B}(0) + O(s^4).$$

Hence the plane through $\alpha(0)$ spanned by $\vec{T}(0)$ and $\vec{N}(0)$, called the osculating plane of α at 0, is the plane through $\alpha(0)$ that best approximates the curve α there. Indeed,

$\alpha(s) = \alpha(0) + s\vec{T}(0) + \frac{k(0)}{2}s^2\vec{N}(0) + O(s^3)$ so distance $(\alpha(s), \text{osculating plane}) = O(s^3)$.

For regular curves $\alpha(t)$ in $\mathbb{R}^3$ not necessarily parametrized by arc length, the curvature and torsion of α at t are defined instead by

$$k(t) = \frac{|\vec{T}'(t)|}{|\alpha'(t)|}, \quad \tau(t) = \frac{\vec{B}'(t)\cdot\vec{N}(t)}{|\alpha'(t)|} = -\frac{\vec{N}'(t)\cdot\vec{B}(t)}{|\alpha'(t)|}.$$

(Normalize by $|\alpha'(t)|$!)

These definitions are invariant under reparametrization of the regular curve α. The Frenet formula now reads

$$\begin{pmatrix} \vec{T}'(t) \\ \vec{N}'(t) \\ \vec{B}'(t) \end{pmatrix} = |\alpha'(t)| \begin{pmatrix} 0 & k(t) & 0 \\ -k(t) & 0 & -\tau(t) \\ 0 & \tau(t) & 0 \end{pmatrix} \begin{pmatrix} \vec{T}(t) \\ \vec{N}(t) \\ \vec{B}(t) \end{pmatrix}.$$

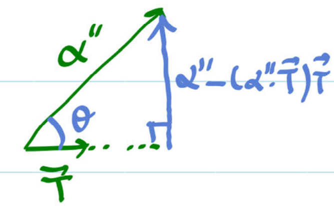
We also have the following alternative formula for the curvature and the torsion of a regular curve $\alpha(t)$ in $\mathbb{R}^3$:

$$k(t) = \frac{|\alpha'(t)\wedge\alpha''(t)|}{|\alpha'(t)|^3} \quad \text{and} \quad \tau(t) = -\frac{\alpha'(t)\cdot(\alpha''(t)\wedge\alpha'''(t))}{|\alpha'(t)\wedge\alpha''(t)|^2}$$

which hold even if $\alpha(t)$ is not parametrized by arc length.

Indeed, to prove $k(t)=\frac{|\alpha'(t)\wedge\alpha''(t)|}{|\alpha'(t)|^3}$ and $\tau(t) = -\frac{\alpha'(t)\cdot(\alpha''(t)\wedge\alpha'''(t))}{|\alpha'(t)\wedge\alpha''(t)|^2}$

we differentiate $\vec{T} = \frac{\alpha'}{|\alpha'|}$ to obtain $\vec{T}' = \frac{\alpha'' - (\alpha''\cdot\vec{T})\vec{T}}{|\alpha'|}$, so



$$|\vec{T}'| = \frac{|\alpha''||\vec{T}|\sin\theta}{|\alpha'|} = \frac{|\alpha''\wedge\vec{T}|}{|\alpha'|} = \frac{|\alpha'\wedge\alpha''|}{|\alpha'|^2} \Rightarrow k = \frac{|\vec{T}'|}{|\alpha'|} = \frac{|\alpha'\wedge\alpha''|}{|\alpha'|^3}.$$

Furthermore, $\vec{N} = \frac{\vec{T}'}{|\vec{T}'|} = \frac{|\alpha'|\vec{T}'}{|\alpha'||\vec{T}'|}$, so

$$\tau = \frac{-\vec{N}'\cdot\vec{B}}{|\alpha'|} = -\frac{1}{|\alpha'|}\left(\frac{1}{|\alpha'||\vec{T}'|}\right)'|\alpha'|\underbrace{\vec{T}'\cdot\vec{B}}_{=|\alpha'|\,k\,\vec{N}\cdot\vec{B}=0} - \frac{1}{|\alpha'|^2|\vec{T}'|}\left(\alpha''-(\alpha''\cdot\vec{T})\vec{T}\right)'\cdot\vec{B}$$

$$= -\frac{1}{|\alpha'|^2|\vec{T}'|}\alpha'''\cdot\vec{B} + \frac{1}{|\alpha'|^2|\vec{T}'|}\left[(\alpha''\cdot\vec{T})'\underbrace{\vec{T}\cdot\vec{B}}_{=0} + (\alpha''\cdot\vec{T})\underbrace{\vec{T}'\cdot\vec{B}}_{=0}\right]$$

$$= -\frac{\alpha'''\cdot(\alpha'\wedge\alpha'')}{|\alpha'\wedge\alpha''|^2}$$ since $\vec{B} = \frac{\alpha'\wedge\alpha''}{|\alpha'\wedge\alpha''|}$ and $|\vec{T}'| = \frac{|\alpha'\wedge\alpha''|}{|\alpha'|^2}$

Exercise. Calculate the curvature and torsion of the curve $\alpha(t) = (t, t^2, t^3)$

Answer: $k(t) = \frac{2(1+9t^2+3t^4)^{1/2}}{(1+4t^2+9t^4)^{3/2}}$, $\tau(t) = \frac{3}{1+9t^2+9t^4}$.

Fundamental theorem of local theory of space curves

Theorem. Given C^∞ functions $k(s) > 0$ and $\tau(s)$, there exists a C^∞ regular curve $\alpha(s)$ parametrized by arc length, so that $k(s)$ and $\tau(s)$ are the curvature and torsion of α at s.
Furthermore, $k(s)$ and $\tau(s)$ determine the image of α up to a rigid motion in $\mathbb{R}^3$ (i.e. a rotation followed by a translation in $\mathbb{R}^3$).

Intuitively, the curvature determines how the curve bends on the osculating plane, and the torsion determines how the curve twists away from the osculating plane. (Proof of theorem omitted).

Isoperimetric inequality for plane curves

Theorem. Let C be a simple closed plane curve with length L ("simple" means C has no self intersection).
Let A be the area of the region bounded by C.
Then $A \le \frac{L^2}{4\pi}$, and equality holds $\Leftrightarrow$ C is a circle.

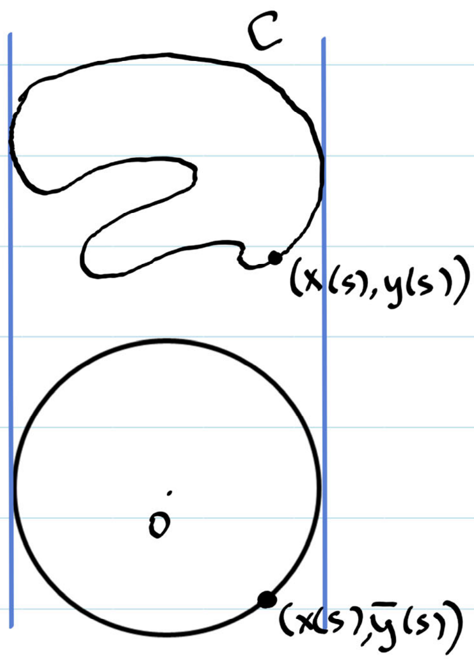
Remark. If C is a simple closed curve in $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus C$ is always the union of two connected open sets, one bounded (called the region bounded by C), one unbounded. This is called the Jordan curve theorem, and while intuitive is quite difficult to prove!

Exercise. How long is the shortest fence you need to enclose some piece of land of area π? Answer: minimum length $= (4\pi \cdot \pi)^{1/2} = 2\pi$.

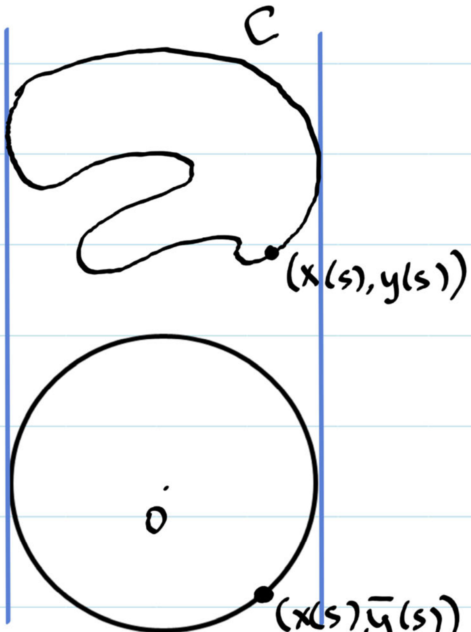
There is also an isoperimetric inequality for hypersurfaces in higher dimensions, which says that spheres in $\mathbb{R}^n$ provide the most efficient way of enclosing a given volume in $\mathbb{R}^n$.

Proof of the isoperimetric inequality for plane curves:

Given C, set up coordinates as follows:



Fit C between two vertical lines,
fit a circle between the two lines
Let 0 be the centre of circle
Let $(x(s), y(s))$ be arc length parametrization of C, $s \in [0, L]$. Find $\bar{y}(s)$ so that $(x(s), \bar{y}(s))$ parametrizes the circle (not necessarily by arc length, and may need to trace through parts of circle more than once).



Let A = area bounded by $C = \int_0^L x(s)y'(s)\,ds$

$\bar{A}$ = area bounded by circle $= -\int_0^L x'(s)\bar{y}(s)\,ds$ Green's formula

Also let r = radius of circle.

Then

$$
\begin{aligned}
2\sqrt{A\pi r^2} &= 2\sqrt{A\bar{A}} \\
&\leq A + \bar{A} \\
&= \int_0^L \left(x(s)y'(s) - x'(s)\bar{y}(s) \right) ds \\
&\leq \left(\int_0^L x(s)^2 + \bar{y}(s)^2 \, ds \right)^{\frac{1}{2}} \left(\int_0^L x'(s)^2 + y'(s)^2 \, ds \right)^{\frac{1}{2}}
\end{aligned}
$$

(Cauchy-Schwarz)

$$
= \sqrt{Lr^2} \cdot \sqrt{L}
$$

$$
\Rightarrow \quad A \leq \frac{L^2}{4\pi}.
$$

If equality holds, then $A = \bar{A} = \pi r^2$ and $\frac{x}{y'} = -\frac{\bar{y}}{x'}$, so $\frac{x^2}{y'^2} = \frac{\bar{y}^2}{x'^2}$, which implies $\frac{x^2}{y'^2} = \frac{\bar{y}^2 + x^2}{x'^2 + y'^2} = \frac{r^2}{1^2} = r^2$, i.e. $x^2 = r^2 y'^2$.

The rest is a little tricky!

We could have set up coordinates with the roles of x and y reversed. So if we had $x^2 = r^2 y'^2$, then we also have $y^2 = r^2 x'^2$. Hence $x^2 + y^2 = r^2(x'^2 + y'^2) = r^2$. This shows that C is a circle of radius r.