Surfaces in $\mathbb{R}^3$

§1. Parametrized surfaces in $\mathbb{R}^3$

Definition. A parametrized surface is a C^∞ mapping

$$\underline{x} : \mathcal{U} \longrightarrow \mathbb{R}^3$$

defined on some open set $\mathcal{U} \subseteq \mathbb{R}^2$.

It is said to be a regular parametrized surface if its differential $d\underline{x}_a : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is injective $\forall a \in \mathcal{U}$.

3×2 matrix of partial derivatives of the 3 components of $\underline{x}$ evaluated at a.

E.g. $\underline{x}(u,v) = (u, v, f(u,v))$ for some C^∞ function $f : \mathcal{U} \to \mathbb{R}$ defines a parametrized surface in $\mathbb{R}^3$. (called the graph of f)

Is $\underline{x} : \mathcal{U} \to \mathbb{R}^3$ a regular parametrized surface? Yes.

Is $d\underline{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix}$ injective everywhere in $\mathcal{U}$? Yes.

If $d\underline{x}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ then $\begin{pmatrix} w_1 \\ w_2 \\ w_1\frac{\partial f}{\partial u} + w_2\frac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Note: We usually write $\underline{x}(u,v) = (x(u,v), y(u,v), z(u,v))$ for $(u,v) \in \mathcal{U}$,

and write

$$\underline{x}_u = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix}, \quad \underline{x}_v = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix}.$$

Then the following are equivalent:

(i) $d\underline{x}$ is injective everywhere on $\mathcal{U}$;

(ii) $\underline{x}_u$ and $\underline{x}_v$ are linearly independent everywhere on $\mathcal{U}$;

(iii) $\underline{x}_u \wedge \underline{x}_v \neq 0$ everywhere on $\mathcal{U}$.

(iv) the 3×2 matrix $\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$ has rank 2 everywhere in $\mathcal{U}$;

(v) Everywhere in $\mathcal{U}$, at least one of the following determinants is $\neq 0$:

$$\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}, \quad \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}, \quad \det\begin{pmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}.$$

E.g. If $\alpha : I \to \mathbb{R}^3$ is a parametrized space curve, then

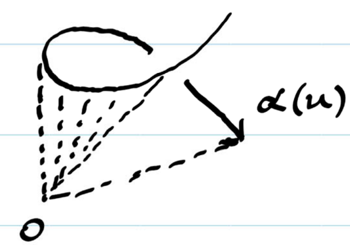
$\underline{x}(u,v) = v\,\alpha(u)$ $\forall (u,v) \in I \times (0,\infty)$ defines a parametrized surface in $\mathbb{R}^3$

Is $\underline{x}$ a regular parametrized surface?

It depends. Note $\underline{x}_u = v\alpha'(u)$ and $\underline{x}_v = \alpha(u)$

So $\underline{x}$ is a regular parametrized surface, if and only if $\alpha'(u)$ is not a multiple of $\alpha(u)$ $\forall u \in I$.

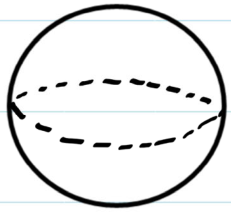
Note α may have self-intersections, in which case $\underline{x}$ is not injective; this is allowed in the definition of a (regular) parametrized surface.



still a regular parametrized surface.

This is sometimes inconvenient; e.g. at self-intersections there can appear to be more than one "tangent plane".

The main objects of study will hence be surfaces without self intersection and non-regular points.

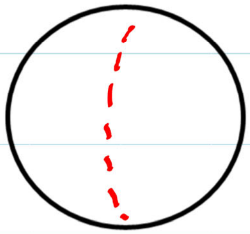


e.g. We certainly want to study the sphere S^2 but it is not possible to find an open set $U \subseteq \mathbb{R}^2$ and a regular parametrized surface $\underline{x}: U \to \mathbb{R}^3$ such that $\underline{x}$ is a bijection between U and S^2. (S^2 minus a point is still simply connected but U minus a point is not)

We can try parametrizing S^2 using spherical coordinates, but we will always miss some points on S^2 if we insist the domain U to be open and the parametrization $\underline{x}$ be injective:

$\underline{Q}$. What about $\underline{x}: (0,2\pi)\times(0,\pi) \to S^2 \subseteq \mathbb{R}^3$ given by

$$\underline{x}(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi) ?$$



It misses a longitude between the poles $(0,0,1)$ and $(0,0,-1)$.

This motivates us to make another (different!) definition.

§2 Regular surfaces in $\mathbb{R}^3$

Definition. A subset $S \subseteq \mathbb{R}^3$ is a regular surface if $\forall p \in S$, $\exists$ open set $V \subseteq \mathbb{R}^3$ containing p and a C^∞ mapping $\underline{x} : \mathcal{U} \to V \cap S$ on an open set $\mathcal{U} \subseteq \mathbb{R}^2$ such that

(i) $\underline{x}$ is a homeomorphism,

i.e. $\underline{x}$ is bijective and $\underline{x}^{-1} : V \cap S \to \mathcal{U}$ is continuous

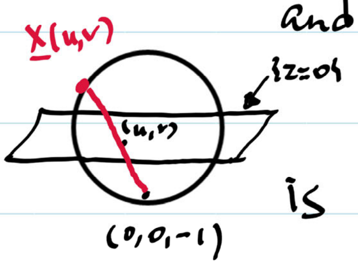
(ii) $d\underline{x}_a : \mathbb{R}^2 \to \mathbb{R}^3$ is injective at every point $a \in \mathcal{U}$.

Question: Are "Regular surfaces" and "regular parametrized surfaces" the same?

No! First, a regular surface S may need a few different maps to be completely parametrized.

Second, a regular surface S cannot have "self-intersections" (or anything close to self intersecting) eg. $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$ is a regular surface.

Indeed, if $p \in S^2 \setminus \{(0,0,-1)\}$, then let $V = \mathbb{R}^2 \times (-1,\infty)$, $V \cap S^2 = S^2 \setminus \{(0,0,-1)\}$, and the stereographic projection $\underline{x} : \mathbb{R}^2 \to V \cap S^2 \subseteq \mathbb{R}^3$ given by



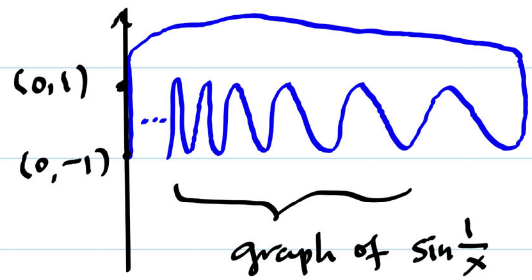
$$\underline{x}(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2} \right)$$

is C^∞, homeomorphism of $\mathbb{R}^2$ onto $V \cap S^2$, and $d\underline{x}$ is injective at every point on $\mathbb{R}^2$. If $p = (0,0,-1)$, let $V = \mathbb{R}^2 \times (-\infty, 1)$, and use the other stereographic projection $\left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2} \right)$ onto $S^2 \setminus \{(0,0,1)\}$ instead.

eg. The product of the figure 8 curve with $\mathbb{R}$ is <u>not</u> a regular surface, because if p is a point of self-intersection on this surface S, then $\nexists$ open set $V \subseteq \mathbb{R}^3$ containing p such that $V \cap S$ is homeomorphic to an open subset of $\mathbb{R}^2$.

(If $\mathcal{U} \subseteq \mathbb{R}^2$ is open and $\underline{x} : \mathcal{U} \to V \cap S$ is continuous bijection, then $\underline{x}^{-1} : V \cap S \to \mathcal{U}$ is <u>not</u> continuous.)

Question. Is the product of the following plane curve with the real line a regular surface in $\mathbb{R}^3$?



No! eg If $p=(0,0,0)$ then there is no open set $V \subseteq \mathbb{R}^3$ containing p such that $V \cap$ this surface is homeomorphic to an open set in $\mathbb{R}^2$.

eg. Graphs of C^∞ functions are regular surfaces:

If $f: \mathcal{U} \to \mathbb{R}$ is C^∞ on some open set $\mathcal{U} \subseteq \mathbb{R}^2$, then $S = \{(x,y,z) \in \mathcal{U} \times \mathbb{R} : z = f(x,y)\}$ is a regular surface in $\mathbb{R}^3$.

Indeed, $\forall p \in S$, take $V = \mathcal{U} \times \mathbb{R}$ open in $\mathbb{R}^3$, and $V \cap S = S$.

The map $\underline{x}: \mathcal{U} \to V \cap S \subseteq \mathbb{R}^3$ given by $\underline{x}(u,v) = (u, v, f(u,v))$ is C^∞, homeomorphism onto $V \cap S = S$, and

$$d\underline{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_u & f_v \end{pmatrix}$$ is injective at every point in $\mathcal{U}$.

e.g. Let $F : \Omega \to \mathbb{R}$ be a C^∞ function on some open set $\Omega \subseteq \mathbb{R}^3$.

If $\alpha \in F(\Omega)$, we say that α is a regular value of F, if $dF(p) \neq (0,0,0) \quad \forall p \in F^{-1}(\alpha)$.

If α is a regular value of F, then $S = F^{-1}(\alpha)$ is a regular surface.

Indeed, let $p \in S$. Without loss of generality, assume $\frac{\partial F}{\partial z}(p) \neq 0$.

Then the implicit function theorem applies: $\exists$ open set $V \subseteq \mathbb{R}^3$ containing p, and a C^∞ function $\phi : \mathcal{U} \to \mathbb{R}$ such that $(u,v,w) \in V \cap S \iff (u,v) \in \mathcal{U}$ and $w = \phi(u,v)$; here $\mathcal{U} = \pi(V)$ and $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the coordinate projection onto the first two variables. Hence the C^∞ map $\underline{x} : \mathcal{U} \to V \cap S \subseteq \mathbb{R}^3$ given by $\underline{x}(u,v) = (u, v, \phi(u,v))$ is well-defined and a homeomorphism, and $d\underline{x} : \mathbb{R}^2 \to \mathbb{R}^3$ is injective on $\mathcal{U}$ as we have verified before.

Question. This provides another proof that S^2 is a regular surface. Why?

Because $S^2 = F^{-1}(0)$ where $F(x,y,z) = x^2 + y^2 + z^2 - 1$, and 0 is a regular value of F.

The C^∞ structure on a regular surface.

Recall the definition of a regular surface S:

It is a subset of $\mathbb{R}^3$, such that $\forall\, p \in S$, $\exists$ open set $V \subseteq \mathbb{R}^3$ containing p, and a C^∞ map $\underline{x} : \mathcal{U} \to V \cap S \subseteq \mathbb{R}^3$ on some open set $\mathcal{U} \subseteq \mathbb{R}^2$, such that

(i) $\underline{x} : \mathcal{U} \to V \cap S$ is a homeomorphism

i.e. $\underline{x}$ is continuous, bijective and $\underline{x}^{-1} : V \cap S \to \mathcal{U}$ is continuous

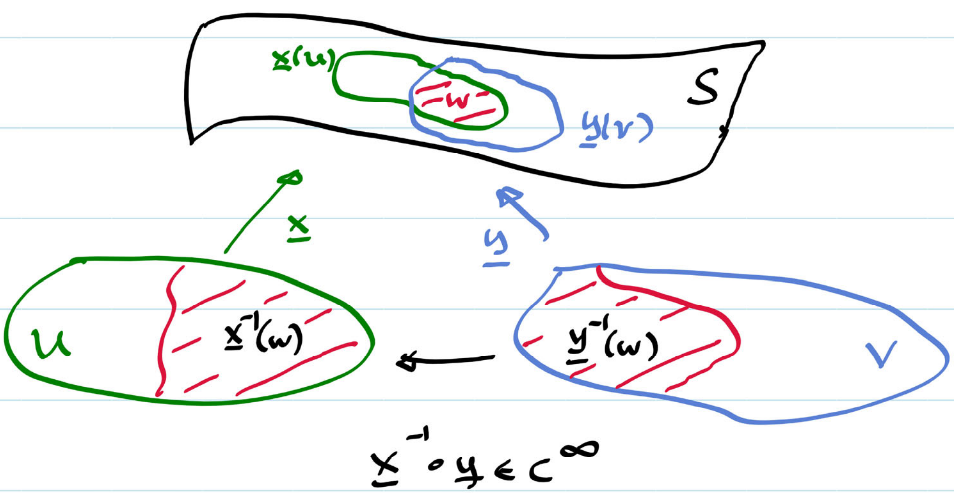
(ii) $d\underline{x} : \mathbb{R}^2 \to \mathbb{R}^3$ is injective everywhere on $\mathcal{U}$.

Note that we only required $\underline{x}^{-1}$ to be continuous; we did not ask it to be differentiable (or C^∞) even though $\underline{x}$ is.

This is because we haven't even defined what it means for a function from $V \cap S$ to $\mathcal{U} \subseteq \mathbb{R}^2$ to be differentiable (or C^∞).

To be able to do so, we need the following theorem.

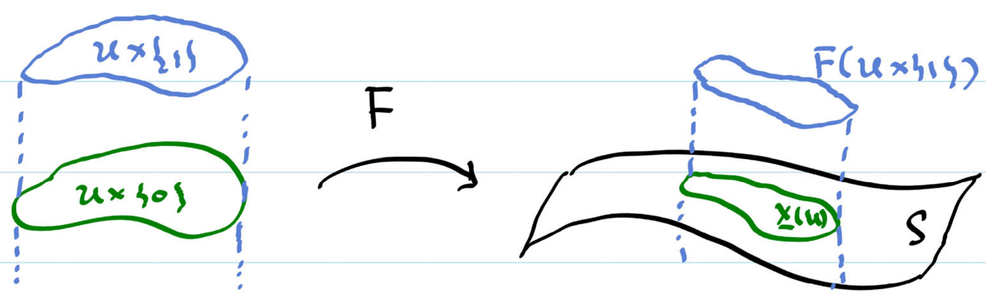
Theorem. Let S be a regular surface in $\mathbb{R}^3$. Suppose $\underline{x}$, $\underline{y}$ are parametrizations of S on open sets $U, V \subseteq \mathbb{R}^2$ as in the definition of regular surfaces. Let $W = \underline{x}(U) \cap \underline{y}(V)$. Then $\underline{x}^{-1}(W)$, $\underline{y}^{-1}(W)$ are open sets in $\mathbb{R}^2$ and $\underline{x}^{-1} \circ \underline{y} : \underline{y}^{-1}(W) \to \underline{x}^{-1}(W)$ is C^∞.



Remark. The main difficulty in proving this theorem is that we do not know yet whether $\underline{x}^{-1}$ extends to a C^∞ map from some open subset of $\mathbb{R}^3$ to U. This is what we need to do. Once done, the chain rule then tells us that $\underline{x}^{-1} \circ \underline{y}$ is C^∞.

Proof. First it is easy to show that $\underline{x}^{-1}(W)$, $\underline{y}^{-1}(W)$ are open sets in $\mathbb{R}^2$.
Indeed, by definition of a regular surface, $\underline{x}(\mathcal{U})$ is the intersection of an open set in $\mathbb{R}^3$ with S, so $\underline{x}(\mathcal{U})$ is open in S.
Similarly, $\underline{y}(V)$ is open in S. Hence $W = \underline{x}(\mathcal{U}) \cap \underline{y}(V)$ is open in S.
Since $\underline{x}$ is continuous on an open set $\mathcal{U}$ of $\mathbb{R}^2$, $\underline{x}^{-1}(W)$ is open in $\mathbb{R}^2$.
Similarly, $\underline{y}^{-1}(W)$ is open in $\mathbb{R}^2$.

Next, let $p \in \underline{x}(\mathcal{U}) \cap \underline{y}(V)$, and we extend $\underline{x}^{-1}$ to some C^∞ function on an open set in $\mathbb{R}^3$ containing p. To do so, write $\underline{x}(u,v) = (x(u,v), y(u,v), z(u,v))$ and $\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\Big|_a \neq 0$ where $\underline{x}(a) = p$. Let $\tilde{\mathcal{U}} = \mathcal{U} \times \mathbb{R}$ and $F: \tilde{\mathcal{U}} \to \mathbb{R}^3$ be defined by $F(u,v,w) = \underline{x}(u,v) + (0,0,w)$ $\forall (u,v,w) \in \tilde{\mathcal{U}}$ (addition carried out in $\mathbb{R}^3$).

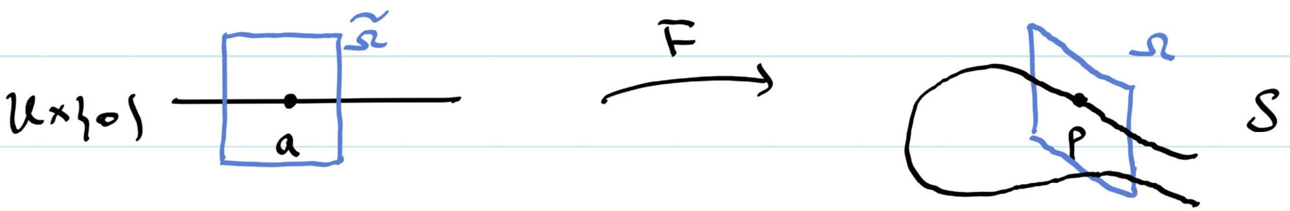


Then since $F(u,v,w) = \underline{x}(u,v) + (0,0,w)$, if $p = \underline{x}(a)$ for some $a \in U$,

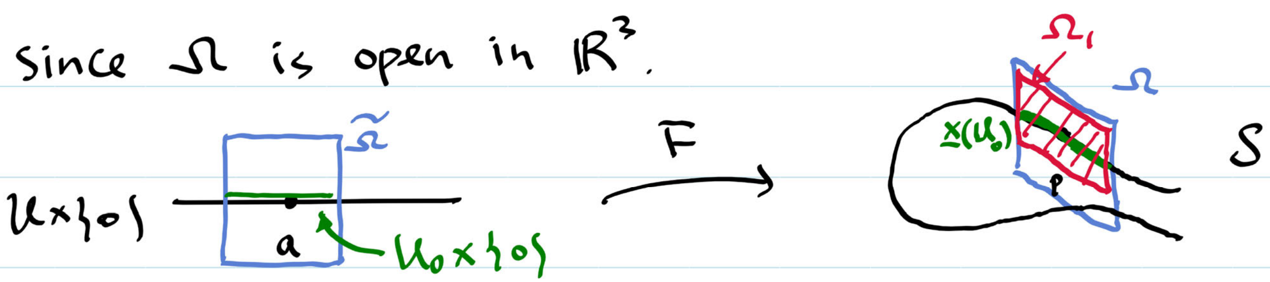
then $\det dF(a,0) = \det \left(d\underline{x}_a \;\middle|\; \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\Bigg|_a \neq 0$

So the inverse function theorem applies:

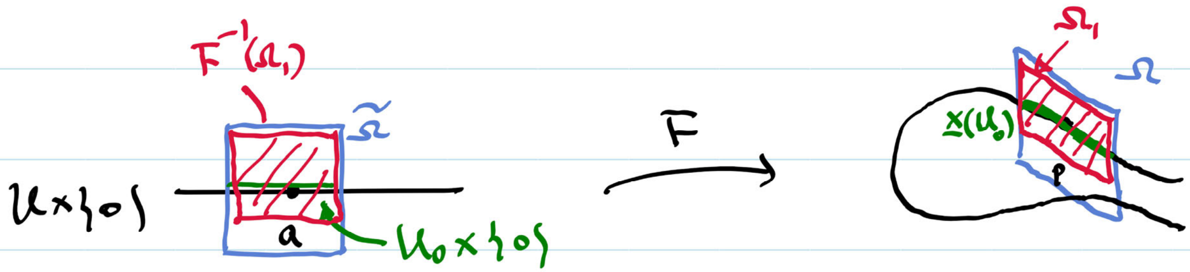
$\exists$ open sets $\Omega \subseteq \mathbb{R}^3$ containing $p = F(a,0)$ and $\tilde{\Omega} \subseteq \tilde{U}$ containing $(a,0)$ such that $F : \tilde{\Omega} \to \Omega$ is C^∞, bijective with a C^∞ inverse $F^{-1} : \Omega \to \tilde{\Omega}$.



Let $U_0 := \{(u,v) \in U : (u,v,0) \in \tilde{\Omega}\}$. It is open in $\mathbb{R}^2$. By continuity of $\underline{x}^{-1}$ in the condition on parametrizations in the definition of a regular surface, $\underline{x}(U_0)$ is an open subset of S. As a result, $\exists$ open set $\Omega_1 \subseteq \mathbb{R}^3$ such that $\underline{x}(U_0) = \Omega_1 \cap S$; we will choose $\Omega_1 \subseteq \Omega$ which is possible since Ω is open in $\mathbb{R}^3$.



Pick Ω_1 to avoid $(\Omega \cap S) \setminus \underline{x}(U_0)$, using continuity of $\underline{x}^{-1}$.

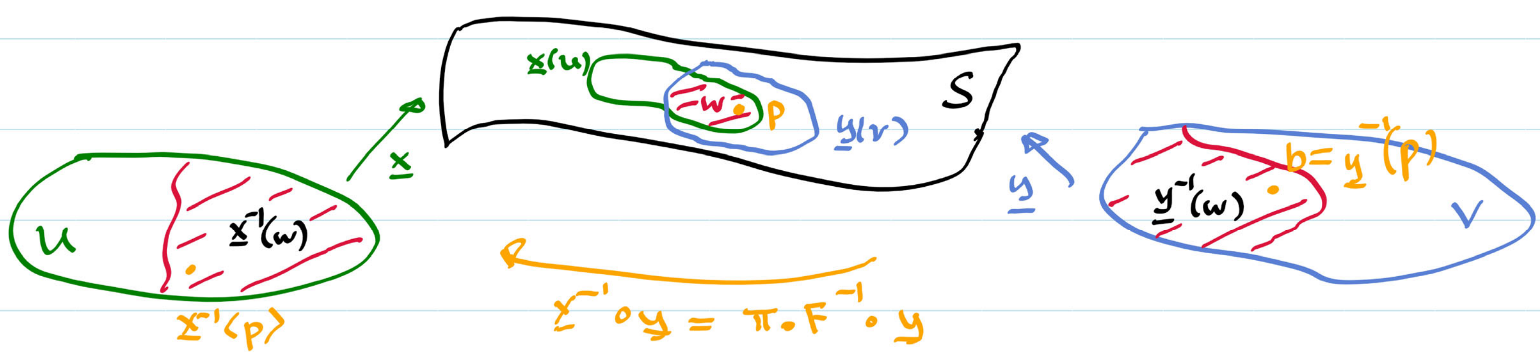


Note that $F:\tilde{\Omega}\to\Omega$ restricts to a C^∞ bijection $F: F^{-1}(\Omega_1)\to\Omega_1$ with C^∞ inverse $F^{-1}:\Omega_1\to F^{-1}(\Omega_1)$

Also, if $\pi:\mathbb{R}^3\to\mathbb{R}^2$ is the coordinate projection onto the first two coordinates, then $\pi\circ F^{-1}\big|_{\underline{x}(\mathcal{U}_0)} = \underline{x}^{-1}\big|_{\underline{x}(\mathcal{U}_0)}$.

Now suppose $p\in W$, and $b=\underline{y}^{-1}(p)$.

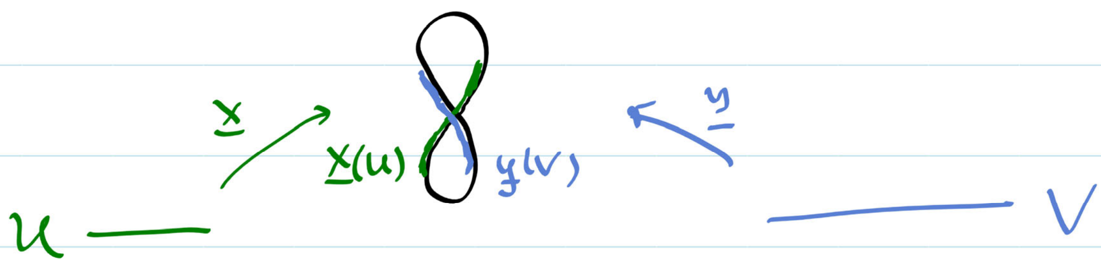
Since $\exists$ open set $V_0\subseteq V$ containing b such that $\underline{y}(V_0)\subseteq\underline{x}(\mathcal{U}_0)$ (just take $V_0=\underline{y}^{-1}(W\cap\underline{x}(\mathcal{U}_0))$ and use continuity of $\underline{y}$) we have $\underline{x}^{-1}\circ\underline{y} = \pi\circ F^{-1}\circ\underline{y}\in C^\infty$ in a neighborhood V_0 of b.



We remark what could go wrong if S is not a regular surface, or if $\underline{x}^{-1}$ is not required to be continuous in the definition of a regular surface.

eg. What if S is the product of the figure 8 curve with $\mathbb{R}$?

Then maybe we have $\underline{x}: U \to S$ and $\underline{y}: V \to S$ as follows



In this case, $W = \underline{x}(U) \cap \underline{y}(V)$ is a line over the point where the figure 8 curve intersects itself, and $\underline{x}^{-1}(W)$, $\underline{y}^{-1}(W)$ are each just a curve in U and V respectively.

They are not open, and the map $\underline{x}^{-1} \circ \underline{y}$ is not defined on an open set.

eg. What if $S =$ product of figure 9 curve with the real line?

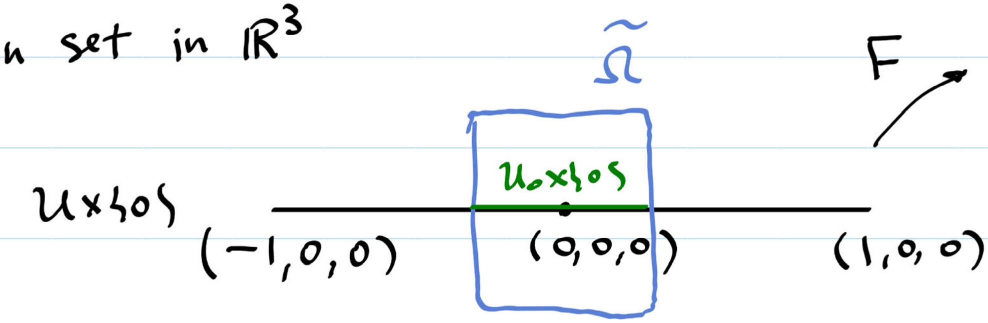
More precisely, let $\alpha : (-1,1) \to \mathbb{R}^2$ be a parametrization of the figure 9 curve, so that $\lim_{t\to 1^-} \alpha(t) = \alpha(0) = (0,0)$.

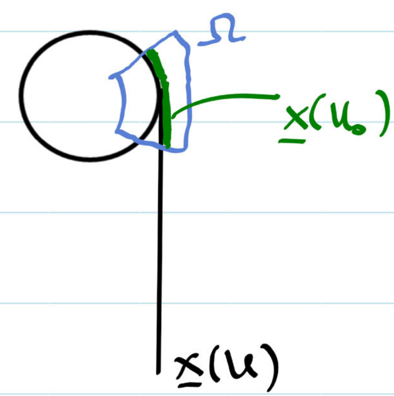
Let $S = \{ (\alpha(u), v) : u \in (-1,1),\ v \in \mathbb{R} \}$.

Let $\mathcal{U} = (-1,1) \times \mathbb{R}$, $\underline{x}(u,v) = (\alpha(u), v)$ for $(u,v) \in \mathcal{U}$. Let $p = (0,0,0)$, and $\tilde{\Omega}$, Ω, $\mathcal{U}_0$ be as in the proof of the theorem.

Then $\underline{x}(\mathcal{U}_0)$ is not open in S; there is no way to choose an open set $\Omega_1 \subseteq \mathbb{R}^3$ such that $\underline{x}(\mathcal{U}_0) = \Omega_1 \cap S$

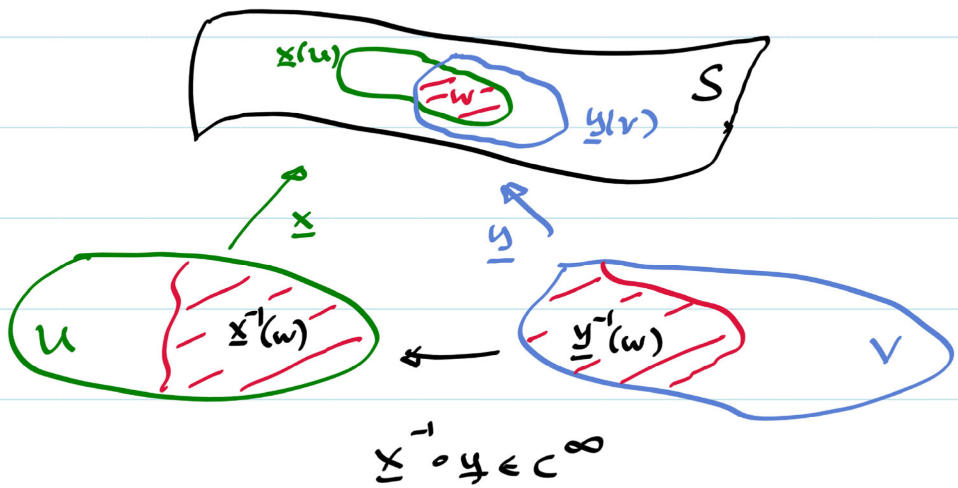
Also $\underline{x}^{-1}$ is discontinuous at p, so there is certainly no way to extend $\underline{x}^{-1}$ to a differentiable map on an open set in $\mathbb{R}^3$ containing p.





Summary

Theorem. Let S be a regular surface in $\mathbb{R}^3$. Suppose $\underline{x}$, $\underline{y}$ are parametrizations of S on open sets $U, V \subseteq \mathbb{R}^2$ as in the definition of regular surfaces. Let $W = \underline{x}(U) \cap \underline{y}(V)$. Then $\underline{x}^{-1}(W), \underline{y}^{-1}(W)$ are open sets in $\mathbb{R}^2$ and $\underline{x}^{-1} \circ \underline{y} : \underline{y}^{-1}(W) \to \underline{x}^{-1}(W)$ is C^∞.



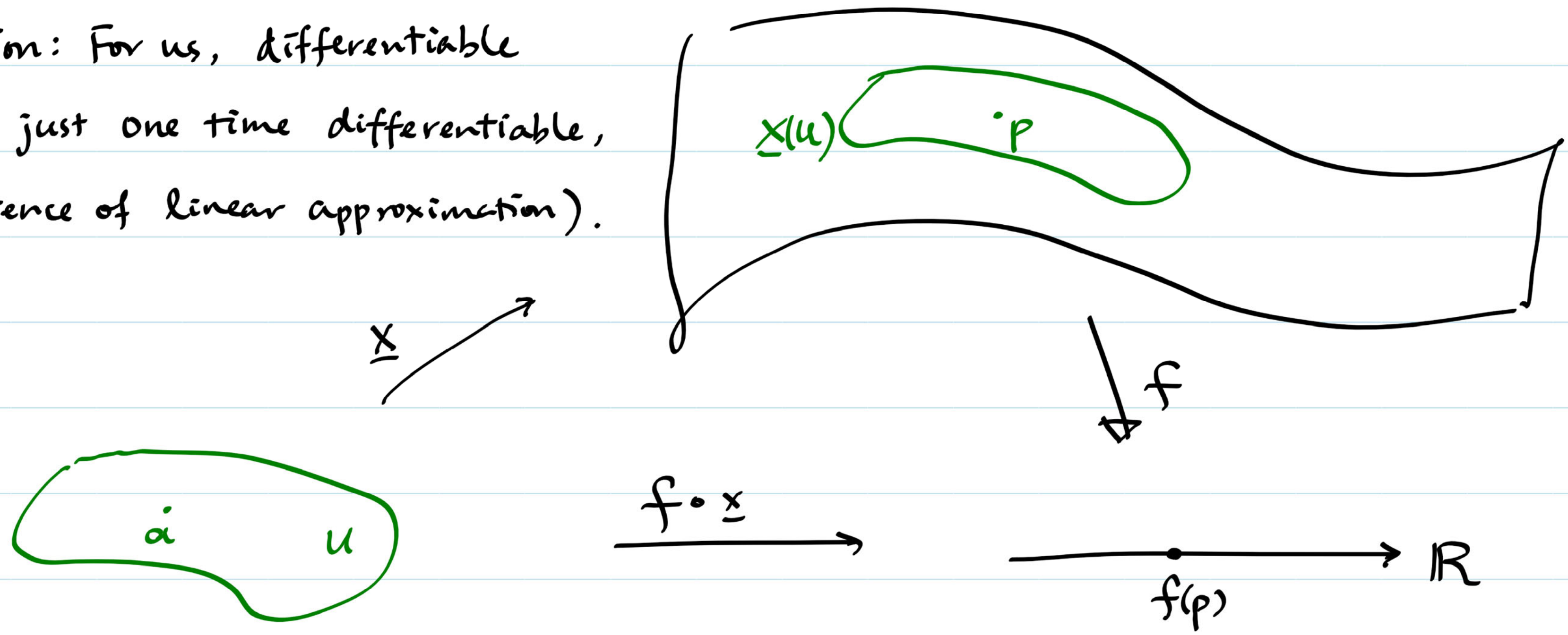
A point p in W can either be described by $\underline{x}^{-1}(p)$ (its $\underline{x}$-coords) or by $\underline{y}^{-1}(p)$ (its $\underline{y}$-coords). The theorem asserts that the change of variables map $\underline{x}^{-1} \circ \underline{y}$ is C^∞ $\rightsquigarrow$ definition of C^∞ functions on S.

Differentiable functions on a regular surface

Let S be a regular surface, $p \in S$, and $f : S \to \mathbb{R}$.

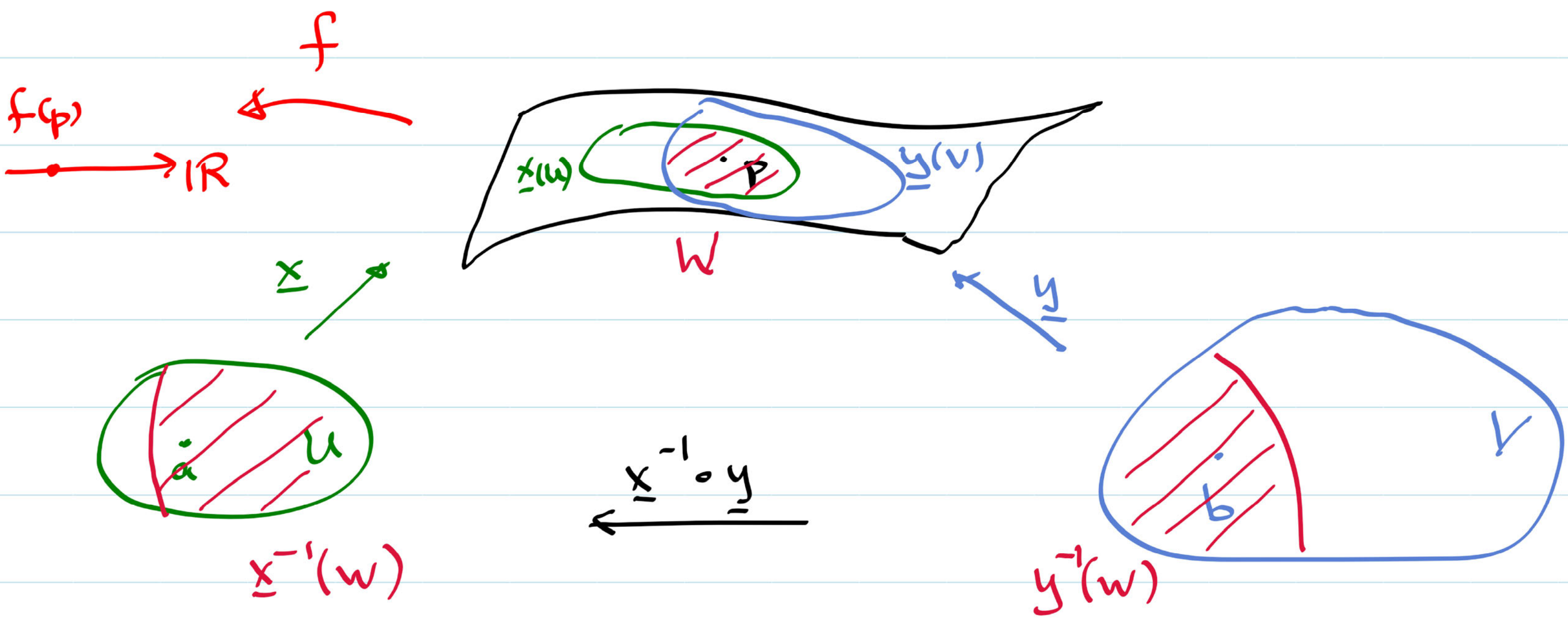
We say f is differentiable (resp. C^∞) at p, if $\exists$ a parametrization $\underline{x} : \mathcal{U} \to S$ on some open set $\mathcal{U} \subseteq \mathbb{R}^2$, such that $\underline{x}(a) = p$ for some $a \in \mathcal{U}$, and such that $f \circ \underline{x} : \mathcal{U} \to \mathbb{R}$ is differentiable (resp. C^∞) at a.

(Convention: For us, differentiable means just one time differentiable, i.e. existence of linear approximation).



If $f \circ \underline{x}$ is differentiable at a for one such parametrization, then for any other parametrizations $\underline{y} : V \to S$ with $\underline{y}(b) = p$ for some $b \in V$, we also have $f \circ \underline{y}$ being differentiable at b.

Indeed, $f \circ \underline{y} = f \circ \underline{x} \circ (\underline{x}^{-1} \circ \underline{y})$ on $\underline{y}^{-1}(W)$ where $W = \underline{x}(U) \cap \underline{y}(V)$ and $\underline{x}^{-1} \circ \underline{y}$ is differentiable on $\underline{y}^{-1}(W)$ by a previous theorem. So differentiability of $f : S \to \mathbb{R}$ at p is independent of which parametrization of S we use near p.



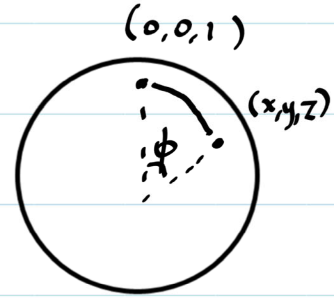
e.g. If $F: \mathbb{R}^3 \to \mathbb{R}$ is differentiable, and S is a regular surface in $\mathbb{R}^3$, then F restricts to a differentiable function $f: S \to \mathbb{R}$. Why?

To see that $f: S \to \mathbb{R}$ is differentiable, let $p \in S$ and $\underline{x}: \mathcal{U} \to S$ be a parametrization with $p \in \underline{x}(\mathcal{U})$.

Then $f \circ \underline{x}: \mathcal{U} \to \mathbb{R}$ is given by $f \circ \underline{x}(u,v) = F(\underline{x}(u,v))$ $\forall (u,v) \in \mathcal{U}$, which is a differentiable function on $\mathcal{U}$

More generally, if $p \in S$, and $f: S \to \mathbb{R}$ extends to a differentiable function on some open set Ω in $\mathbb{R}^3$ containing p, then f is differentiable at p.

eg. If $S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$ the unit sphere, and $f: S \to \mathbb{R}$ is such that $f(x,y,z)$ = length of shortest path from (x,y,z) to $(0,0,1)$, then later we will see that $f(x,y,z)$ = angle ϕ between $(0,0,1)$ and $(x,y,z) = \cos^{-1} z$. Where is f differentiable?



Answer: f is differentiable (indeed C^∞) at every $p \in S \setminus \{(0,0,1),(0,0,-1)\}$, and f is <u>not</u> differentiable at $(0,0,\pm 1)$ (because $\cos^{-1}(\pm\sqrt{1-x^2-y^2})$ is not differentiable at $(x,y)=(0,0)$).

Tangent vectors and tangent spaces

If S is a regular surface and $p \in S$, the set of vectors

$$\left\{ \alpha'(0) : \begin{array}{l} \alpha : (-1,1) \to S \text{ is a } C^\infty \text{ curve on } S \\ \text{and } \alpha(0) = p \end{array} \right\}$$

is called the tangent plane to S at p, written $T_p(S)$.

A moment's reflection shows that this is a vector space: if $\underline{x} : U \to S$ is a parametrization with $p = \underline{x}(a)$, then $T_p(S)$ is spanned by $\underline{x}_u$ and $\underline{x}_v$ at $(u,v) = a$.

This is because if $\alpha : (-1,1) \to S$ is a curve on S then $\tilde{\alpha} := \underline{x}^{-1} \circ \alpha : (-1,1) \to \mathbb{R}^2$ is a plane curve, and $\alpha'(0) = d\underline{x}_a(\tilde{\alpha}'(0))$ is a linear combination of $\underline{x}_u$ and $\underline{x}_v$ at a. If S is a regular surface and $p \in S$, then a vector $v \in T_pS$ is called a tangent vector to S at p.

The tangent plane to a regular surface S at p can be computed once we determine the normal vector to $T_p(S)$. If $\underline{x}: U \to S$ is a parametrization with $p = \underline{x}(a)$ where $a \in U$, then a normal vector to $T_p(S)$ is $\underline{x}_u \wedge \underline{x}_v$ evaluated at a.

Question. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be differentiable. Find the equation of the tangent plane to $S = \{(x,y,z) \in \mathbb{R}^3 : z = f(x,y)\}$ at $(x_0, y_0, f(x_0,y_0))$.

(Answer: $A(x-x_0) + B(y-y_0) + C(z - f(x_0,y_0)) = 0$ where $A = \partial_x f(x_0,y_0)$, $B = \partial_y f(x_0,y_0)$, $C = -1$.)

eg. If $F: \mathbb{R}^3 \to \mathbb{R}$ is C^∞ and 0 is a regular value of F, then $\forall p \in S := F^{-1}(0)$, $\nabla F(p)$ is a normal to $T_p(S)$.

This is because if $v \in T_p(S)$, say $v = \alpha'(0)$ for some $\alpha: (-1,1) \to S$ with $\alpha(0) = p$, then $F(\alpha(t)) = 0$ $\forall t \in (-1,1)$, so differentiating we have $0 = \nabla F(p) \cdot \alpha'(0) = \nabla F(p) \cdot v$, which shows $\nabla F(p) \perp T_p(S)$.

$\rightsquigarrow$ Can now determine the equation of tangent plane $T_p(S)$.

If $f: S \to \mathbb{R}$ is differentiable at $p \in S$ and $V \in T_p(S)$, the directional derivative $df_p(v)$ is defined to be $\frac{d}{dt} f(\alpha(t))\Big|_{t=0}$ where $\alpha: (-1,1) \to S$ is any curve with $\alpha(0) = p$ and $\alpha'(0) = v$.

Note $df_p(v)$ is defined independent of choice of α, and $df_p : T_p(S) \to \mathbb{R}$ is linear.

e.g. If $f: S^2 = \{x^2+y^2+z^2=1\} \to \mathbb{R}$ is given by $f(x,y,z) = \cos^{-1} z$, then for $p = (1,0,0)$, $e_1 = (0,1,0) \in T_p S$, $e_2 = (0,0,1) \in T_p S$, we have $df_p(e_1) = 0$, $df_p(e_2) = -1$ (For $df_p(e_1)$, consider eg. $\alpha_1(t) = (\cos t, \sin t, 0)$; for $df_p(e_2)$, consider eg $\alpha_2(t) = (\sqrt{1-t^2}, 0, t)$).

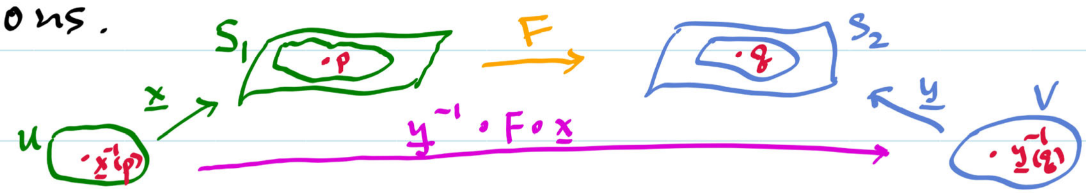
If $f: S \to \mathbb{R}$ is differentiable at $p \in S$, then $\exists$ a unique tangent vector in $T_p(S)$, called the gradient of f at p and denoted $\nabla f(p)$, such that $df_p(v) = v \cdot \nabla f(p)$ $\forall v \in T_p(S)$.

Note: The uniqueness assertion is only true because we require $\nabla f(p) \in T_p(S)$. In fact, if N_p is a normal vector to $T_p(S)$, then still $df_p(v) = (\nabla f(p) + N_p) \cdot v$ $\forall v \in T_p(S)$.

eg. In the above example, $\nabla f(1,0,0) = (0,0,-1)$.

Differentiable maps between regular surfaces ; the differential

Similarly, we define differentiable maps between regular surfaces.
Let S_1, S_2 be regular surfaces, $F: S_1 \to S_2$, $p \in S_1$ and $q = F(p)$. Let $\underline{x}: U \to S_1$, $\underline{y}: V \to S_2$ be parametrizations of S_1, S_2 respectively. Then F is said to be differentiable (resp. C^∞) at p, if and only if $\underline{y}^{-1} \circ F \circ \underline{x}$ is differentiable (resp. C^∞) at $\underline{x}^{-1}(p)$.
Again whether or not $\underline{y}^{-1} \circ F \circ \underline{x}$ is differentiable is independent of the choice of parametrizations.



If $F: S_1 \to S_2$ is differentiable at some $p \in S_1$, its differential $(dF)_p : T_p S_1 \to T_{F(p)} S_2$ is defined by

$$(dF)_p(v) = \frac{d}{dt}\Big|_{t=0} F(\alpha(t))$$

where $\alpha(t)$ is any curve in S_1 with $\alpha(0) = p$ and $\alpha'(0) = v$.

If $F: S_1 \to S_2$ is differentiable at p, and $\underline{x}: U \to S_1$, $\underline{y}: V \to S_2$ are parametrizations with $p \in \underline{x}(U)$ and $F(p) \in \underline{y}(V)$, then $\underline{y}^{-1} \circ F \circ \underline{x}$ is given (on $\underline{x}^{-1}(\underline{x}(U) \cap \underline{y}(V))$) by

$$\underline{y}^{-1} \circ F \circ \underline{x}\,(u,v) = (F_1(u,v), F_2(u,v))$$

for some functions F_1 and F_2 that are differentiable at $a := \underline{x}^{-1}(p)$, and if $v \in T_p S_1$, $v = v_1 \underline{x}_u + v_2 \underline{x}_v$ at a, then $(dF)_p(v)$ is given by

$$(dF)_p(v) = w_1 \underline{y}_{\tilde{u}} + w_2 \underline{y}_{\tilde{v}} \quad \text{at} \quad \underline{y}^{-1}(F(p))$$

where

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial u}(a) & \frac{\partial F_1}{\partial v}(a) \\ \frac{\partial F_2}{\partial u}(a) & \frac{\partial F_2}{\partial v}(a) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Using the chain rule in 2-dimensions, if $F: S_1 \to S_2$ is differentiable at $p \in S_1$, and $G: S_2 \to S_3$ is differentiable at $F(p) \in S_2$, then $G \circ F: S_1 \to S_3$ is differentiable at p, and $d(G \circ F)_p = dG_{F(p)} \circ dF_p$ on $T_p(S_1)$.

If $F: S_1 \to S_2$ is differentiable at every $p \in S_1$, we simply say F is differentiable on S_1.

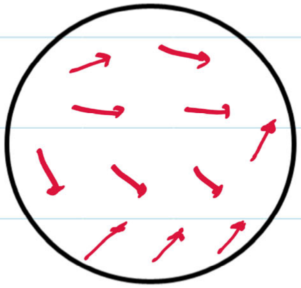
If $F: S_1 \to S_2$ is C^∞ and bijective, then by inverse function theorem (applied to $\underline{y}^{-1} \circ F \circ \underline{x}$ on an open set in $\mathbb{R}^2$), its inverse $F^{-1}: S_2 \to S_1$ is also C^∞, and F is said to be a diffeomorphism between S_1 and S_2. In this case S_1 and S_2 are said to be diffeomorphic to each other.

e.g. If $\mathcal{U}$ is an open set in $\mathbb{R}^2$, $g, h: \mathcal{U} \to \mathbb{R}$ are C^∞, and S_1, S_2 are the regular surfaces $S_1 = \{(x,y,g(x,y)) \in \mathbb{R}^3 : (x,y) \in \mathcal{U}\}$ and $S_2 = \{(x,y,h(x,y)) \in \mathbb{R}^3 : (x,y) \in \mathcal{U}\}$, then S_1 and S_2 are diffeomorphic to each other, with a diffeomorphism given by $F: S_1 \to S_2$, defined by $F(x,y,g(x,y)) = (x,y,h(x,y))\ \forall (x,y) \in \mathcal{U}$.

Indeed, if $\underline{x}: \mathcal{U} \to S_1$ is the parametrization $\underline{x}(u,v) = (u,v,g(u,v))$ and $\underline{y}: \mathcal{U} \to S_2$ is the parametrization $\underline{y}(u,v) = (u,v,h(u,v))$, then $\underline{y}^{-1} \circ F \circ \underline{x} : \mathcal{U} \to \mathcal{U}$ is given by $\underline{y}^{-1} \circ F \circ \underline{x}(u,v) = (u,v)$, so if $W = v_1 \underline{x}_u + v_2 \underline{x}_v$ at $p = \underline{x}(u,v)$, then $(dF)_p(W) = v_1 \underline{y}_u + v_2 \underline{y}_v$ at $\underline{y}(u,v)$, because $\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Vector fields on regular surfaces

Definition. A vector field on a regular surface S is a map that associates to each point $p \in S$ to a tangent vector $w(p) \in T_p(S)$.



e.g. The velocity of wind on Earth defines a vector field on the surface of Earth.

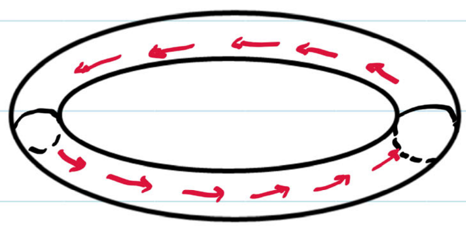
e.g. If S is the torus parametrized by

$\underline{x}(\theta, \phi) = ((2+\cos\phi)\cos\theta, (2+\cos\phi)\sin\theta, \sin\phi)$,

and for $p = \underline{x}(\theta, \phi)$ we define

$$w(p) = \underline{x}_\theta \in T_p(S),$$

then w defines a vector field on S.



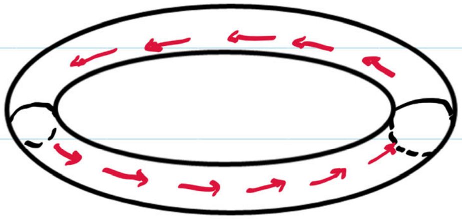
e.g. If $N(p)$ denotes a unit normal vector to $T_p(S)$ $\forall p \in S$, then N is <u>not</u> a vector field on S

Let S be a regular surface, $p \in S$, and w be a vector field on S. We say the vector field w is differentiable at p if $w(\underline{x}(u,v)) = a(u,v)\,\underline{x}_u + b(u,v)\,\underline{x}_v$ where a and b are differentiable functions of (u,v) near $\underline{x}^{-1}(p)$, and $\underline{x}(u,v)$ is any local parametrization of S near p. (It does not matter which local parametrization is used; if the coefficients a and b are differentiable in one parametrization, then so are the coefficients in other parametrizations.) Similarly, the vector field w is said to be C^∞ on an open set $\mathcal{U}$ on S, if $w(\underline{x}(u,v)) = a(u,v)\,\underline{x}_u + b(u,v)\,\underline{x}_v$ where a, b are C^∞ functions on $\mathcal{U}$.

e.g. The vector field $\underline{x}_\theta$ on the torus parametrized by

$$\underline{x}(\theta,\phi) = ((2+\cos\phi)\cos\theta,\ (2+\cos\phi)\sin\theta,\ \sin\phi)$$

on the previous page is C^∞ on the torus.



Example. Let $N=(0,0,1)$ and $S=(0,0,-1)$ be the north and south poles of the sphere $S^2=\{(x,y,z): x^2+y^2+z^2=1\}$. Let $\underline{x}:\mathbb{R}^2\to S^2\setminus\{S\}$ and $\underline{y}:\mathbb{R}^2\to S^2\setminus\{N\}$ be parametrizations by stereographic projections

$$\underline{x}(u,v)=\left(\frac{2u}{1+u^2+v^2},\ \frac{2v}{1+u^2+v^2},\ \frac{1-u^2-v^2}{1+u^2+v^2}\right),\quad \underline{y}(\xi,\eta)=\left(\frac{2\xi}{1+\xi^2+\eta^2},\ \frac{2\eta}{1+\xi^2+\eta^2},\ \frac{-1+\xi^2+\eta^2}{1+\xi^2+\eta^2}\right).$$

Is the vector field $w(p)=\begin{cases} u\,\underline{x}_u(u,v)+v\,\underline{x}_v(u,v) & \text{if } p=\underline{x}(u,v) \\ 0 & \text{if } p=S \end{cases}$ C^∞ on S^2?

Answer. Yes! Indeed, if $\xi(u,v):=\frac{u}{u^2+v^2}$ and $\eta(u,v):=\frac{v}{u^2+v^2}$ for $(u,v)\in\mathbb{R}^2\setminus\{(0,0)\}$, then $\underline{y}(\xi(u,v),\eta(u,v))=\underline{x}(u,v)$ $\forall (u,v)\in\mathbb{R}^2\setminus\{(0,0)\}$. So on $S^2\setminus\{N,S\}$, chain rule gives

$$\begin{cases} \underline{x}_u = \frac{\partial\xi}{\partial u}\underline{y}_\xi + \frac{\partial\eta}{\partial u}\underline{y}_\eta \\ \underline{x}_v = \frac{\partial\xi}{\partial v}\underline{y}_\xi + \frac{\partial\eta}{\partial v}\underline{y}_\eta \end{cases}$$

where the vectors on the left hand side are evaluated at (u,v), and the vectors on the right are evaluated at $(\xi(u,v),\eta(u,v))$. Since $\frac{\partial\xi}{\partial u}=-\frac{\partial\eta}{\partial v}=\frac{v^2-u^2}{(u^2+v^2)^2}$, $\frac{\partial\xi}{\partial v}=\frac{\partial\eta}{\partial u}=-\frac{2uv}{(u^2+v^2)^2}$, this shows

$$(u\underline{x}_u+v\underline{x}_v)\Big|_{\underline{x}(u,v)} = -\frac{u}{u^2+v^2}\underline{y}_\xi-\frac{v}{u^2+v^2}\underline{y}_\eta = -(\xi\,\underline{y}_\xi+\eta\,\underline{y}_\eta)\Big|_{\underline{y}(\xi(u,v),\eta(u,v))}.$$

Hence $w(p)=-(\xi\,\underline{y}_\xi+\eta\,\underline{y}_\eta)$ if $p=\underline{y}(\xi,\eta)$ for some $(\xi,\eta)\in\mathbb{R}^2\setminus\{(0,0)\}$.
The same holds at $p=\underline{y}(0,0)=S$. Hence w is C^∞ on $S^2\setminus\{N\}$.

Remark on notations. In more advanced courses, if $\underline{x}: U \to \underline{x}(U) \subseteq S$ is a local parametrization of a surface S, then the vector fields $\underline{x}_u$ and $\underline{x}_v$ are often written as $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ without explicit reference to $\underline{x}$. This notation is chosen, because if f is a function defined on $\underline{x}(U)$, then for $p = \underline{x}(a)$ with $a \in U$, we have $df_p(\underline{x}_u) = \frac{\partial F}{\partial u}(a)$ and $df_p(\underline{x}_v) = \frac{\partial F}{\partial v}(a)$, where $F(u,v) := f(\underline{x}(u,v))$. Often people even identify F with f, and write $df_p(\underline{x}_u) = \frac{\partial f}{\partial u}(a)$, $df_p(\underline{x}_v) = \frac{\partial f}{\partial v}(a)$. Hence by identifying the vector fields $\underline{x}_u$ and $\underline{x}_v$ with the directional derivatives they induce, people write $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ for $\underline{x}_u$ and $\underline{x}_v$. This is sometimes convenient: e.g. we saw that if $\underline{x}(u,v) = \underline{y}(\xi,\eta)$ where ξ, η are implicit functions of u and v defined by this equation, then
$$\begin{cases} \underline{x}_u = \frac{\partial \xi}{\partial u} \underline{y}_\xi + \frac{\partial \eta}{\partial v} \underline{y}_\eta \\ \underline{x}_v = \frac{\partial \xi}{\partial v} \underline{y}_\xi + \frac{\partial \eta}{\partial v} \underline{y}_\eta \end{cases}.$$
The same equality of vector fields can be written in this new notation as
$$\begin{cases} \frac{\partial}{\partial u} = \frac{\partial \xi}{\partial u} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial v} \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial v} = \frac{\partial \xi}{\partial v} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial v} \frac{\partial}{\partial \eta} \end{cases}$$
which is just the chain rule when we changed variables $(u,v) \mapsto (\xi(u,v), \eta(u,v))$.

More generally, suppose $\underset{\sim}{x} : \mathcal{U} \to \underset{\sim}{x}(\mathcal{U}) \subseteq S_1$ and $\underset{\sim}{y} : V \to \underset{\sim}{y}(V) \subseteq S_2$ are local parametrizations of regular surfaces S_1 and S_2 respectively. If $F : \underset{\sim}{x}(\mathcal{U}) \to \underset{\sim}{y}(V)$ is differentiable, and (u,v) are coordinates on $\mathcal{U}$, (ξ,η) are coordinates on V, then writing $(\xi(u,v), \eta(u,v)) = \underset{\sim}{y}^{-1} \circ F \circ \underset{\sim}{x}(u,v)$, we have

$$\begin{cases} dF_p\left(\frac{\partial}{\partial u}\right) = \frac{\partial \xi}{\partial u}(a)\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial u}(a)\frac{\partial}{\partial \eta} \\ dF_p\left(\frac{\partial}{\partial v}\right) = \frac{\partial \xi}{\partial v}(a)\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial v}(a)\frac{\partial}{\partial \eta} \end{cases}$$

whenever $p = \underset{\sim}{x}(a)$ and $a \in \mathcal{U}$.

By linearity we recover $dF_p\left(\alpha\frac{\partial}{\partial u} + \beta\frac{\partial}{\partial v}\right)$ for any $\alpha\frac{\partial}{\partial u} + \beta\frac{\partial}{\partial v} \in T_p(S)$.

e.g. If $S_1 = \{(x,y,z) : \frac{x^2}{3^2} + \frac{y^2}{4^2} + \frac{z^2}{5^2} = 1\}$ and $S_2 = \mathbb{S}^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$, then for $F : S_1 \to S_2$ given by $F(x,y,z) = \left(\frac{x}{3}, \frac{y}{4}, \frac{z}{5}\right)$ and $p = (0,0,5) \in S_1$, we may parametrize neighborhoods of p and $F(p) = (0,0,1)$ by $\underset{\sim}{x}(u,v) = \left(u, v, 5\sqrt{1 - \frac{u^2}{3^2} - \frac{v^2}{4^2}}\right)$, $\underset{\sim}{y}(\xi,\eta) = \left(\xi, \eta, \sqrt{1-\xi^2-\eta^2}\right)$, in which case $\underset{\sim}{y}^{-1} \circ F \circ \underset{\sim}{x}(u,v) = \left(\frac{u}{3}, \frac{v}{4}\right)$, so $dF_p\left(\alpha\frac{\partial}{\partial u} + \beta\frac{\partial}{\partial v}\right) = \frac{\alpha}{3}\frac{\partial}{\partial \xi} + \frac{\beta}{4}\frac{\partial}{\partial \eta}$, i.e. $dF_p(\alpha,\beta,0) = \left(\frac{\alpha}{3}, \frac{\beta}{4}, 0\right)$.

Caution: We defined what it means for vector fields to be differentiable, but we haven't discussed how to differentiate a differentiable vector field! More to come ...