The first fundamental form

Let S be a regular surface in $\mathbb{R}^3$.

The dot product in the ambient $\mathbb{R}^3$ gives rise to a bilinear form that describes the inner product of any two vectors in T_pS as p varies over S: we define, for $p \in S$ and $v, w \in T_p(S)$, that

$$\langle v, w \rangle_p := v \cdot w \qquad \text{(dot product in } \mathbb{R}^3\text{)},$$

and the corresponding quadratic form

$$I_p(v) := \langle v, v \rangle_p \quad \forall p \in S, \ v \in T_p(S),$$

is called the first fundamental form on S. Note that the first fundamental form I_p determines the bilinear form $\langle \cdot, \cdot \rangle_p$ via

$$\langle v, w \rangle_p = \frac{1}{4}\left[I_p(v+w) - I_p(v-w) \right] \quad \forall p \in S, \ v, w \in T_p(S).$$

So the first fundamental form on a regular surface S allows us to calculate:

① Length of a tangent vector $v \in T_p(S)$: $|v| := I_p(v)^{\frac{1}{2}}$

② Angle between two tangent vectors $v, w \in T_p(S)$: $\theta = \cos^{-1}\left(\frac{\langle v, w\rangle_p}{|v|\,|w|}\right)$

③ Length of a curve $\alpha : I \to S$: $\text{Length}(\alpha) = \int_I |\alpha'(t)|\, dt$

④ Area of a region in S: If $\underline{x} : U \to S$ is a parametrization and $R \subseteq U$, then the area of $\underline{x}(R)$ is $\iint_R |\underline{x}_u \wedge \underline{x}_v|\, du\,dv$

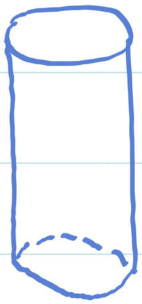
Question: Can you show that the area is independent of the parametrization?

Answer: If $\underline{y} : V \to S$ is another parametrization with $\underline{y}(R') = \underline{x}(R)$, and $\underline{y}(\xi, \eta) = \underline{x}(u,v)$, i.e. $(\xi, \eta) = \underline{y}^{-1} \circ \underline{x}(u,v)$, then $\begin{cases} \underline{x}_u = \xi_u \underline{y}_\xi + \eta_u \underline{y}_\eta \\ \underline{x}_v = \xi_v \underline{y}_\xi + \eta_v \underline{y}_\eta \end{cases}$

$\Rightarrow \underline{x}_u \wedge \underline{x}_v = (\xi_u \eta_v - \xi_v \eta_u)\, \underline{y}_\xi \wedge \underline{y}_\eta$, and $d\xi\, d\eta = \left|\det\begin{pmatrix} \xi_u & \xi_v \\ \eta_u & \eta_v \end{pmatrix}\right| du\,dv$, so

$\int_R |\underline{x}_u \wedge \underline{x}_v|\, du\,dv = \int_{R'} |\underline{y}_\xi \wedge \underline{y}_\eta|\, d\xi\, d\eta$, as desired.

Exercise. Find the first fundamental form of the cylinder parametrized by $\underline{x}(u,v) = (\cos u, \sin u, v)$, $u \in (0, 2\pi)$, $v \in \mathbb{R}$.



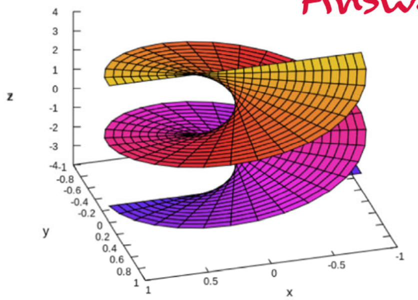
Answer: $E = 1$, $F = 0$, $G = 1$ since

$$\underline{x}_u = (-\sin u, \cos u, 0) \text{ and } \underline{x}_v = (0, 0, 1)$$

$$\Rightarrow E = \langle \underline{x}_u, \underline{x}_u \rangle = 1, \quad F = \langle \underline{x}_u, \underline{x}_v \rangle = 0, \quad G = \langle \underline{x}_v, \underline{x}_v \rangle = 1.$$

Exercise. Find the first fundamental form of the helicoid parametrized by $\underline{x}(u,v) = (v\cos u, v\sin u, u)$, $u \in \mathbb{R}$, $v \in (0, \infty)$.

Answer: $E = v^2 + 1$, $F = 0$, $G = 1$ since



$$\underline{x}_u = (-v\sin u, v\cos u, 1) \text{ and } \underline{x}_v = (\cos u, \sin u, 0)$$

$$\Rightarrow E = \langle \underline{x}_u, \underline{x}_u \rangle = v^2 + 1, \quad F = \langle \underline{x}_u, \underline{x}_v \rangle = 0, \quad G = \langle \underline{x}_v, \underline{x}_v \rangle = 1.$$

(Picture source: Wikipedia)

Exercise. Find the first fundamental form on part of the xy plane in $\mathbb{R}^3$ parametrized by $\underline{x}(r, \theta) = (r\cos\theta, r\sin\theta, 0)$, $\theta \in (0, 2\pi)$, $r \in (0, \infty)$.

Answer: $E = 1$, $F = 0$, $G = r^2$ (so $ds^2 = dr^2 + r^2 d\theta^2$).

We collect some key properties about the family of bilinear forms $\{\langle\cdot,\cdot\rangle_p : p \in S\}$:

① If $p \in S$, then the map $\langle\cdot,\cdot\rangle_p : T_p(S) \times T_p(S) \to \mathbb{R}$ is

$$(v, w) \longmapsto \langle v, w\rangle_p$$

(i) bilinear, i.e.

$$\langle av_1 + bv_2, w\rangle_p = a\langle v_1, w\rangle_p + b\langle v_2, w\rangle_p \quad \forall v_1, v_2, w \in T_p(S),\ a, b \in \mathbb{R}$$

$$\langle v, aw_1 + bw_2\rangle_p = a\langle v, w_1\rangle_p + b\langle v, w_2\rangle_p \quad \forall v, w_1, w_2 \in T_p(S),\ a, b \in \mathbb{R}$$

(ii) symmetric, i.e. $\langle v, w\rangle_p = \langle w, v\rangle_p \quad \forall v, w \in T_p(S)$; and

(iii) positive definite, i.e.

$$\langle v, v\rangle_p \geq 0 \quad \forall v \in T_p(S), \text{ and } \langle v, v\rangle_p = 0 \iff v = 0 \in T_p(S).$$

(Equivalently, the matrix $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is positive definite).

② If v, w are C^∞ vector fields on S, then the function $S \to \mathbb{R}$ given by $p \mapsto \langle v(p), w(p)\rangle_p$ is C^∞ on S

In more advanced courses, e.g. in general relativity, such a smoothly varying positive definite inner product on the tangent spaces (not necessarily coming from an ambient $\mathbb{R}^n$) is called a Riemannian metric.

Sometimes it is customary to define the following functions of $p \in S$:

$$g_{11} = \langle \mathbf{x}_u, \mathbf{x}_u \rangle \quad , \quad g_{12} = \langle \mathbf{x}_u, \mathbf{x}_v \rangle \quad , \quad g_{21} = \langle \mathbf{x}_v, \mathbf{x}_u \rangle \quad \text{and} \quad g_{22} = \langle \mathbf{x}_v, \mathbf{x}_v \rangle.$$

Then $g_{11} = E$, $g_{12} = g_{21} = F$, and $g_{22} = G$,

So the first fundamental form can then be written as

$$ds^2 = g_{11}\, du^2 + 2 g_{12}\, du\, dv + g_{22}\, dv^2,$$

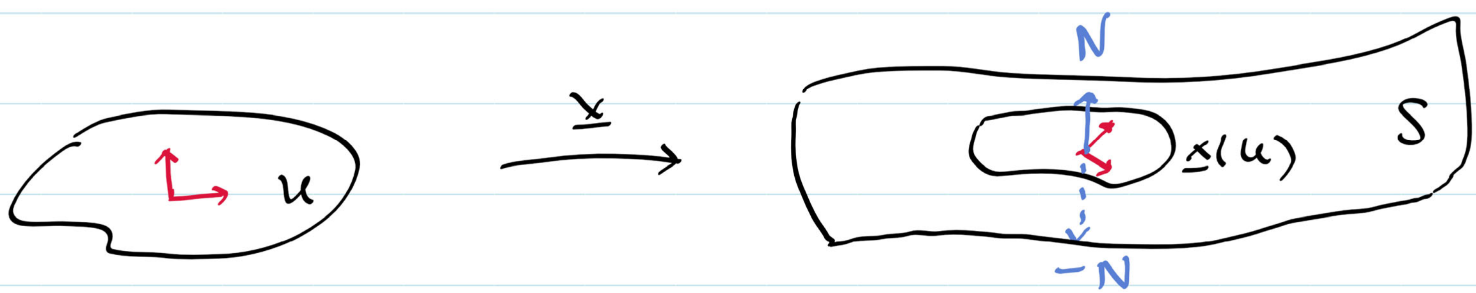
and now we can conveniently write

$$\langle w, w \rangle_p = \sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij}(p)\, w_i w_j \qquad \text{if } w = w_1 \mathbf{x}_u + w_2 \mathbf{x}_v \in T_p(S).$$

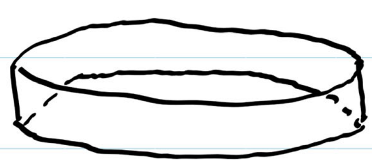
Later in this course: we study isometries, i.e. maps between regular surfaces that preserve the first fundamental form. They automatically preserve the bilinear form $\langle \cdot , \cdot \rangle_p$, because the latter is determined by I_p as we have seen earlier.

Orientable vs Non-orientable surfaces

Let S be a regular surface in $\mathbb{R}^3$, and $\underline{x} : \mathcal{U} \to \underline{x}(\mathcal{U}) \subseteq S$ be a local parametrization. Then $\underline{x}(\mathcal{U})$ has "2 sides", corresponding to the two continuous choices of unit normal $\pm \dfrac{\underline{x}_u \wedge \underline{x}_v}{\|\underline{x}_u \wedge \underline{x}_v\|}$ on $\underline{x}(\mathcal{U})$:

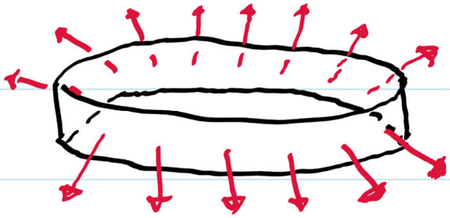


On the other hand, it may not be possible to define a global continuous unit normal on S. eg. compare



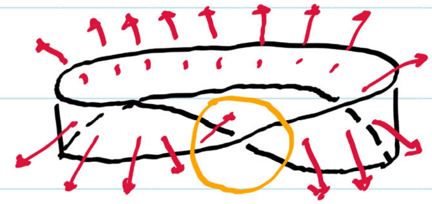
cylinder

Möbius strip



Cylinder

local unit normals match



Möbius strip

Bad matching of local unit normal $\rightarrow$ discontinuity

Definition. A regular surface S in $\mathbb{R}^3$ is said to be orientable, if a continuous choice of unit normal exists on S.

Proposition. Let S be a regular surface in $\mathbb{R}^3$. Then S is orientable, if and only if there exist local parametrizations $\underline{x}_i : U_i \rightarrow \underline{x}(U_i) \subseteq S$ such that $\bigcup_i \underline{x}_i(U_i) = S$ and $\forall i, j$, the change of coordinates map $(u_i, v_i) \longmapsto (u_j, v_j)$ given by $(u_j, v_j) = \underline{x}_j^{-1} \circ \underline{x}_i (u_i, v_i)$ satisfies
$$\det \begin{pmatrix} \frac{\partial u_i}{\partial u_j} & \frac{\partial u_i}{\partial v_j} \\ \frac{\partial v_i}{\partial u_j} & \frac{\partial v_i}{\partial v_j} \end{pmatrix} > 0 .$$

Sketch of proof:

Cover S by $\underline{x}_i(U_i)$ where $\underline{x}_i : U_i \to \underline{x}_i(U_i) \subseteq S$ are local parametrizations. Define

$$N_i := \frac{(\underline{x}_i)_{u_i} \wedge (\underline{x}_i)_{v_i}}{\|(\underline{x}_i)_{u_i} \wedge (\underline{x}_i)_{v_i}\|} \quad \text{on } \underline{x}_i(U_i);$$

it is a continuous unit normal on $\underline{x}_i(U_i)$.

If the change of coordinates $(u_i, v_i) \longmapsto (u_j, v_j)$ have positive Jacobian determinants, then $N_i = N_j$ on $\underline{x}_i(U_i) \cap \underline{x}_j(U_j)$ $\forall i \neq j$, and we obtain a global, continuous choice of unit normal on S.

Conversely, if there is a global, continuous choice of unit normal N on S, then on $\underline{x}_i(U_i)$, either $N = N_i$ or $N = -N_i$.

If $N = N_i$ on $\underline{x}_i(U_i)$, we define $\tilde{\underline{x}}_i(u_i, v_i) = \underline{x}_i(u_i, v_i)$;

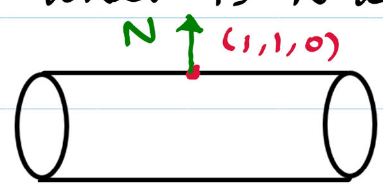
if $N = -N_i$ on $\underline{x}_i(U_i)$, we define $\tilde{\underline{x}}_i(u_i, v_i) = \underline{x}_i(v_i, u_i)$.

In either case, the images of $\tilde{\underline{x}}_i$ parametrizes $\underline{x}_i(U_i)$, and the change of coordinates $\tilde{\underline{x}}_j \circ \tilde{\underline{x}}_i$ have positive Jacobian determinants $\forall i, j$.

If S is an orientable regular surface in $\mathbb{R}^3$, an orientation of S is given by (one of the two) global, continuous choice of unit normals on S.
If S is orientable and an orientation of S is specified, then we say S is an oriented regular surface.

eg. If $F: \mathbb{R}^3 \to \mathbb{R}$ is C^∞ and 0 is a regular value of F, then $F^{-1}(0)$ is orientable, and one orientation of S is given by the (global, continuous) unit normal $N := \dfrac{\nabla F}{\|\nabla F\|}$ on S.

eg. If the cylinder $S = \{(x,y,z) \in \mathbb{R}^3 : y^2+z^2=1\}$ is oriented by the outward unit normal, say N, then what is N at $p=(1,1,0)$?



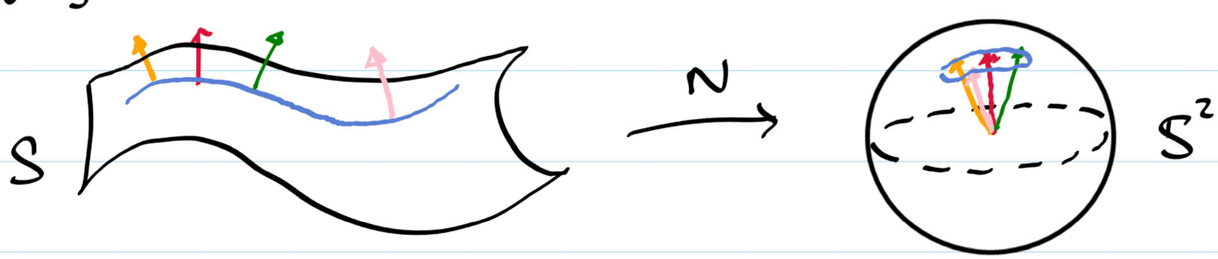
Answer: $N(p) = (0,1,0)$.

The Gauss map and the second fundamental form

Let S be an oriented regular surface in $\mathbb{R}^3$. Then we have a global, continuous unit normal $N : S \to S^2$. This map from S into S^2 is called the Gauss map. It tells us how S curves around in $\mathbb{R}^3$.

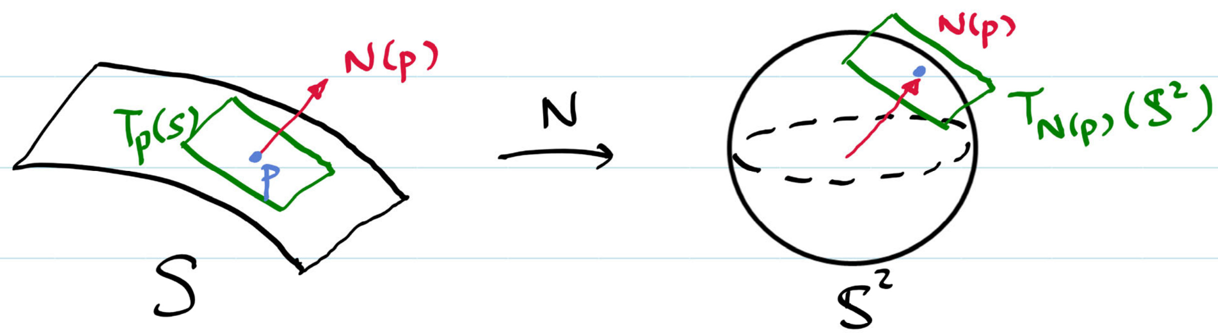
e.g. If S is a plane in $\mathbb{R}^3$, then $N : S \to S^2$ is a constant map.

This suggests that a notion of curvature might be defined, by studying how N varies over the surface S



So let's study the differential dN of the Gauss map N.

If $p \in S$, then $dN_p : T_p(S) \to T_{N(p)}(\mathbb{S}^2)$.



Observation: $T_{N(p)}(\mathbb{S}^2)$ is always parallel to $T_p(S)$. Why?

Because both are normal to $N(p)$.

Convention: We will always identify $T_p(S)$ with $T_{N(p)}(\mathbb{S}^2)$, and think of dN_p as a linear map of $T_p(S)$ into itself, i.e. $dN_p : T_p(S) \to T_p(S)$ $\quad \forall p \in S$.

Definitions. The Gaussian curvature of S at p is the determinant of $-dN_p$, denoted $K(p)$. The mean curvature of S at p is half the trace of $-dN_p$, denoted $H(p)$. ($K(p)$, $H(p)$ measure the "size" of dN_p. Large $K(p)$ & $H(p) \rightsquigarrow S$ very curved at p.)

note sign convention

More explicitly: let $\{e_1, e_2\}$ be a basis of $T_p(S)$. Write

$$\begin{cases} dN_p(e_1) = a_{11}e_1 + a_{12}e_2 \\ dN_p(e_2) = a_{21}e_1 + a_{22}e_2 \end{cases} \qquad \text{—— } (*)$$

Then $K(p) = a_{11}a_{22} - a_{12}a_{21}$, $H(p) = -\dfrac{a_{11}+a_{22}}{2}$.

Question: What is a good basis $\{e_1, e_2\}$ of $T_p(S)$?

Sub Question 1: Can we choose $\{e_1, e_2\}$ so that the matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is in its simplest form? eg. Can we diagonalize dN_p? Yes!

Key idea: $T_p(S)$ carries a natural inner product $\langle \cdot, \cdot \rangle_p$ coming from the first fundamental form (in our case, the restriction of the Euclidean inner product from the ambient $\mathbb{R}^3$), and $dN_p : T_p(S) \to T_p(S)$ is self-adjoint with respect to this inner product, i.e. the matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ defined by $(*)$ is real symmetric as long as $\{e_1, e_2\}$ is orthonormal with respect to $\langle \cdot, \cdot \rangle_p$. So $dN_p : T_p(S) \to T_p(S)$ is diagonalizable: $\exists$ choice of orthonormal $\{e_1, e_2\} \in T_p(S)$ so that $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is diagonal.

Sub Question 2: Can we choose a natural basis $\{e_1, e_2\}$ of $T_p(S)$ so that $dN_p(e_1)$ and $dN_p(e_2)$ are easy to compute?

Maybe $e_1 = \underline{x}_u$ and $e_2 = \underline{x}_v$ at $a = \underline{x}^{-1}(p)$ where $\underline{x} : U \to \underline{x}(U) \subseteq S$ is a local parametrization with $p \in \underline{x}(U)$?

Answer: In that case, $dN_p(e_1) = dN_p(\underline{x}_u)$ and $dN_p(e_2) = dN_p(\underline{x}_v)$, but $dN_p(\underline{x}_u) = \frac{\partial}{\partial u}(N(\underline{x}(u,v)))\Big|_{(u,v)=a}$ which is often abbreviated by saying $dN_p(\underline{x}_u) = N_u$. Similarly $dN_p(\underline{x}_v) = N_v$.

And if we express $\begin{cases} dN_p(\underline{x}_u) = a_{11}\underline{x}_u + a_{12}\underline{x}_v \\ dN_p(\underline{x}_v) = a_{21}\underline{x}_u + a_{22}\underline{x}_v, \end{cases}$ i.e. $\begin{pmatrix} dN_p(\underline{x}_u) \\ dN_p(\underline{x}_v) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \underline{x}_u \\ \underline{x}_v \end{pmatrix}$,

then multiplying by $(\underline{x}_u \ \underline{x}_v)$ at p on the right, we obtain

$$\begin{pmatrix} \langle dN_p(\underline{x}_u), \underline{x}_u \rangle_p & \langle dN_p(\underline{x}_u), \underline{x}_v \rangle_p \\ \langle dN_p(\underline{x}_v), \underline{x}_u \rangle_p & \langle dN_p(\underline{x}_v), \underline{x}_v \rangle_p \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \langle \underline{x}_u, \underline{x}_u \rangle_p & \langle \underline{x}_u, \underline{x}_v \rangle_p \\ \langle \underline{x}_v, \underline{x}_u \rangle_p & \langle \underline{x}_v, \underline{x}_v \rangle_p \end{pmatrix}$$

i.e. $\begin{pmatrix} \langle N_u, \underline{x}_u \rangle_p & \langle N_u, \underline{x}_v \rangle_p \\ \langle N_v, \underline{x}_u \rangle_p & \langle N_v, \underline{x}_v \rangle_p \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}$. We know how to compute the first fundamental form $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ and its inverse.

So to compute $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ using $\begin{pmatrix} \langle N_u, x_u \rangle_p & \langle N_u, x_v \rangle_p \\ \langle N_v, x_u \rangle_p & \langle N_v, x_v \rangle_p \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}$,

it remains to compute the matrix on the left hand side.

We claim that $\begin{cases} \langle N_u, x_u \rangle_p = -\langle N, x_{uu} \rangle_p \ , & \langle N_u, x_v \rangle_p = -\langle N, x_{vu} \rangle_p \\ \langle N_v, x_u \rangle_p = -\langle N, x_{uv} \rangle_p \ , & \langle N_v, x_v \rangle_p = -\langle N, x_{vv} \rangle_p . \end{cases}$

Indeed, if $a = x^{-1}(p)$, then these follow by taking partial derivatives in u and v of the following equations, and then evaluate at $(u,v)=a$:

$$\begin{cases} \langle N, x_u \rangle_{x(u,v)} = 0 \\ \langle N, x_v \rangle_{x(u,v)} = 0 \end{cases} \quad \forall (u,v)$$

This gives us a computationally efficient way to evaluate $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

It is worth observing that since $x_{uv} = x_{vu}$ (mixed partial derivatives commute since $\underline{x}$ is C^∞), we have $\langle N_u, x_v \rangle_p = \langle N_v, x_u \rangle_p$,

so the matrix $\begin{pmatrix} \langle N_u, x_u \rangle_p & \langle N_u, x_v \rangle_p \\ \langle N_v, x_u \rangle_p & \langle N_v, x_v \rangle_p \end{pmatrix} = \begin{pmatrix} \langle dN_p(x_u), x_u \rangle_p & \langle dN_p(x_u), x_v \rangle_p \\ \langle dN_p(x_v), x_u \rangle_p & \langle dN_p(x_v), x_v \rangle_p \end{pmatrix}$

is real symmetric. This shows $\forall w, \tilde{w} \in T_p(S)$, $\langle dN_p(w), \tilde{w} \rangle_p = \langle w, dN_p(\tilde{w}) \rangle_p$.

Indeed, if $w = b_{11}\underline{x}_u + b_{12}\underline{x}_v$ and $\tilde{w} = b_{21}\underline{x}_u + b_{22}\underline{x}_v$ at p, then

$$\langle dN_p(w), \tilde{w}\rangle_p = \langle b_{11}\, dN_p(\underline{x}_u) + b_{12}\, dN_p(\underline{x}_v),\ b_{21}\underline{x}_u + b_{22}\underline{x}_v\rangle_p$$

$$= \begin{pmatrix} b_{11} & b_{12}\end{pmatrix} \begin{pmatrix} \langle dN_p(\underline{x}_u), \underline{x}_u\rangle_p & \langle dN_p(\underline{x}_u), \underline{x}_v\rangle_p \\ \langle dN_p(\underline{x}_v), \underline{x}_u\rangle_p & \langle dN_p(\underline{x}_v), \underline{x}_v\rangle_p \end{pmatrix} \begin{pmatrix} b_{21} \\ b_{22}\end{pmatrix}$$

and similarly

$$\langle w, dN_p(\tilde{w})\rangle_p = \begin{pmatrix} b_{21} & b_{22}\end{pmatrix} \begin{pmatrix} \langle dN_p(\underline{x}_u), \underline{x}_u\rangle_p & \langle dN_p(\underline{x}_u), \underline{x}_v\rangle_p \\ \langle dN_p(\underline{x}_v), \underline{x}_u\rangle_p & \langle dN_p(\underline{x}_v), \underline{x}_v\rangle_p \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{12}\end{pmatrix}.$$

So the symmetry of the matrix in red, we have $\langle dN_p(w), \tilde{w}\rangle_p = \langle w, dN_p(\tilde{w})\rangle_p$.

In particular, the quadratic form $\mathrm{II}_p(w) := -\langle dN_p(w), w\rangle_p \quad \forall w \in T_p(S)$ is real and symmetric, and it is called the second fundamental form of S at p. If $w = w_1\underline{x}_u + w_2\underline{x}_v$ at p, then $\mathrm{II}_p(w) = e w_1^2 + 2f w_1 w_2 + g w_2^2$

where

$$e := -\langle dN_p(\underline{x}_u), \underline{x}_u\rangle_p = -\langle N_u, \underline{x}_u\rangle_p = \langle \underline{x}_{uu}, N\rangle_p$$

$$f := -\langle dN_p(\underline{x}_u), \underline{x}_v\rangle_p = -\langle dN_p(\underline{x}_v), \underline{x}_u\rangle_p = -\langle N_v, \underline{x}_u\rangle_p = -\langle N_u, \underline{x}_v\rangle_p = \langle \underline{x}_{uv}, N\rangle_p$$

$$g := -\langle dN_p(\underline{x}_v), \underline{x}_v\rangle_p = -\langle N_v, \underline{x}_v\rangle_p = \langle \underline{x}_{vv}, N\rangle_p$$

note sign convention

are the coefficients of the second fundamental form.

Furthermore, the equality $\langle dN_p(w), \tilde{w} \rangle_p = \langle w, dN_p(\tilde{w}) \rangle_p \quad \forall\, w, \tilde{w} \in T_p(S)$ shows that $dN_p : T_p(S) \to T_p(S)$ is self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_p$ as we have promised earlier. As a result, if $\{e_1, e_2\}$ is any orthonormal basis of $T_p(S)$ with respect to $\langle \cdot, \cdot \rangle_p$, and

$$\begin{cases} dN_p(e_1) = a_{11}e_1 + a_{12}e_2 \\ dN_p(e_2) = a_{21}e_1 + a_{22}e_2 \end{cases} \quad \text{———} \quad (*),$$

then by orthonormality of $\{e_1, e_2\}$, $a_{ij} = \langle dN_p(e_i), e_j \rangle_p$ for $i, j \in \{1, 2\}$, and by self-adjointness of $dN_p : T_p(S) \longrightarrow T_p(S)$, we have $a_{12} = a_{21}$ (because $a_{12} = \langle dN_p(e_1), e_2 \rangle_p = \langle e_1, dN_p(e_2) \rangle_p = a_{21}$). This says $-\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is then a real symmetric matrix, and hence $\exists$ orthogonal 2×2 matrix Q such that $Q\begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix} Q^t$ is a diagonal matrix $\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$. If $Q = (\tilde{e}_1\ \tilde{e}_2)$ where $\tilde{e}_1, \tilde{e}_2$ are the columns of Q, then $(*)$ becomes $\begin{cases} -dN_p(\tilde{e}_1) = k_1\tilde{e}_1 \\ -dN_p(\tilde{e}_2) = k_2\tilde{e}_2 \end{cases}$ ($\tilde{e}_1, \tilde{e}_2 \in T_p(S)$ are eigenvectors of $-dN_p$).

So $-dN_p$ is represented by a diagonal matrix $\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ in the orthonormal basis $\{\tilde{e}_1, \tilde{e}_2\}$ of $T_p(S)$, and $K(p) = k_1 k_2$, $H(p) = \dfrac{k_1 + k_2}{2}$.

The second fundamental form, Gaussian and mean curvatures: Computation

Let's summarise our discussion of the second fundamental form.

Let S be an oriented regular surface with continuous unit normal N on S.

The second fundamental form is the real, symmetric quadratic form $\mathbb{I}_p$ on $T_p(S)$, defined by $\mathbb{I}_p(w) = -\langle dN_p(w), w\rangle$, $\forall w \in T_p(S)$

Some authors drop this minus sign

If $\underline{x}: U \to \underline{x}(U) \subseteq S$ is a local parametrization and $p = \underline{x}(a)$ with $a \in U$, then for $w = w_1 \underline{x}_u + w_2 \underline{x}_v \in T_p(S)$, we have

$$\mathbb{I}_p(w) = e w_1^2 + 2f w_1 w_2 + g w_2^2 \left(= (w_1 \; w_2) \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right)$$

where $e = \langle \underline{x}_{uu}, N\rangle_p$, $f = \langle \underline{x}_{uv}, N\rangle_p = \langle \underline{x}_{vu}, N\rangle_p$, $g = \langle \underline{x}_{vv}, N\rangle_p$.

In fact, if $w = w_1 \underline{x}_u + w_2 \underline{x}_v$ and $\tilde{w} = \tilde{w}_1 \underline{x}_u + \tilde{w}_2 \underline{x}_v$ at p, then we also have $\langle dN_p(w), \tilde{w}\rangle_p = e w_1 \tilde{w}_1 + f(w_1 \tilde{w}_2 + w_2 \tilde{w}_1) + g w_2 \tilde{w}_2 = \langle w, dN_p(\tilde{w})\rangle_p$

If $\begin{cases} dN_p(\underline{x}_u) = a_{11} \underline{x}_u + a_{12} \underline{x}_v \\ dN_p(\underline{x}_v) = a_{21} \underline{x}_u + a_{22} \underline{x}_v \end{cases}$ at p, then $-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}$

where $E = \langle \underline{x}_u, \underline{x}_u\rangle_p$, $F = \langle \underline{x}_u, \underline{x}_v\rangle_p$, $G = \langle \underline{x}_v, \underline{x}_v\rangle_p$ $\rightsquigarrow$ compute $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

{Some authors drop this minus sign}

Once we compute $-\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ at p, we may compute the Gaussian curvature $K(p)$ and the mean curvature $H(p)$ by taking the determinant and half the trace of this matrix, i.e.

Some authors drop this "half"

$$K(p) = a_{11}a_{22} - a_{12}a_{21}, \quad H(p) = -\frac{a_{11}+a_{22}}{2}.$$

This is well-defined independent of the choice of $\underline{x}: U \to S$, because the determinant and trace of the linear map $dN_p: T_p(S) \to T_p(S)$ is well-defined independent of the basis with respect to which you express this linear map. Recalling $-\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$, we have

$$K(p) = \frac{eg - f^2}{EG - F^2}, \quad H(p) = \frac{eG - 2fF + gE}{2(EG - F^2)}.$$

If $\{\underline{x}_u, \underline{x}_v\}$ is orthonormal at p, i.e. if $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is the identity matrix, then this simplifies because then we simply have $-\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$ (a real symmetric matrix); in that case, $K(p) = eg - f^2$, $H(p) = \frac{e+g}{2}$.

Note: If we had picked the opposite orientation and chose $-N$ instead of N, then K remains unchanged, but H changes sign!

Example. Let $S = \{(x,y,z) : z = Ax + By\}$ be a plane in $\mathbb{R}^3$. Find the second fundamental form of S, and compute the Gaussian and the mean curvatures at every $p \in S$, with respect to the normal that has a negative third component.

Solution. Parametrize S via $\underline{x}(u,v) = (u, v, Au + Bv)$ $\forall (u,v) \in \mathbb{R}^2$.

Then $\underline{x}_u = (1, 0, A)$, $\underline{x}_v = (0, 1, B)$,

and $\underline{x}_{uu} = \underline{x}_{uv} = \underline{x}_{vv} = (0, 0, 0)$. So

$E = \langle \underline{x}_u, \underline{x}_u \rangle = 1 + A^2$, $F = \langle \underline{x}_u, \underline{x}_v \rangle = AB$, $G = \langle \underline{x}_v, \underline{x}_v \rangle = 1 + B^2$,

$$\frac{\underline{x}_u \wedge \underline{x}_v}{\|\underline{x}_u \wedge \underline{x}_v\|} = \frac{(-A, -B, 1)}{\sqrt{A^2+B^2+1}} \quad \text{so} \quad N = -\frac{\underline{x}_u \wedge \underline{x}_v}{\|\underline{x}_u \wedge \underline{x}_v\|} = \frac{(A, B, -1)}{\sqrt{A^2+B^2+1}}.$$

$e = \langle \underline{x}_{uu}, N \rangle = 0$, $f = \langle \underline{x}_{uv}, N \rangle = 0$, $g = \langle \underline{x}_{vv}, N \rangle = 0$

So if $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$, then

$a_{11} = a_{12} = a_{21} = a_{22} = 0$, and $K(p) = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = 0$,

$H(p) = -\frac{1}{2} \operatorname{trace}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = 0.$

Example. Let $S = \{(x,y,z) : x^2+y^2 = R^2\}$ be a cylinder of radius R. Find the second fundamental form of S, and compute the Gaussian and the mean curvatures at every $p \in S$, with respect to the outward unit normal of S.

Solution. Parametrize S by $\underline{x}(u,v) = (R\cos u, R\sin u, v)$.

Then $\underline{x}_u = (-R\sin u, R\cos u, 0)$, $\underline{x}_v = (0,0,1)$

and $\underline{x}_{uu} = (-R\cos u, -R\sin u, 0)$, $\underline{x}_{uv} = \underline{x}_{vv} = (0,0,0)$.

$E = \langle \underline{x}_u, \underline{x}_u \rangle = R^2$, $F = \langle \underline{x}_u, \underline{x}_v \rangle = 0$, $G = \langle \underline{x}_v, \underline{x}_v \rangle = 1$,

$\dfrac{\underline{x}_u \wedge \underline{x}_v}{\|\underline{x}_u \wedge \underline{x}_v\|} = (\cos u, \sin u, 0)$ is outward pointing, so $N = (\cos u, \sin u, 0)$.

$e = \langle \underline{x}_{uu}, N \rangle = -R$, $f = \langle \underline{x}_{uv}, N \rangle = 0$, $g = \langle \underline{x}_{vv}, N \rangle = 0$

If $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$, then $-\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$

so taking determinant and half trace, we get $K = 0$, $H = -\frac{1}{2R}$.

Example. Let $S = \{(x,y,z) : x^2+y^2+z^2 = R^2\}$ be a sphere of radius R. Find the second fundamental form of S, and compute the Gaussian and the mean curvatures at every $p \in S$ with respect to the outward unit normal

Solution. Parametrize part of S by $\underline{x}(u,v) = R(\cos u \cos v, \sin u \cos v, \sin v)$, $u \in (0, 2\pi)$, $v \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Then $\underline{x}_u = R(-\sin u \cos v, \cos u \cos v, 0)$, $\underline{x}_v = R(-\cos u \sin v, -\sin u \sin v, \cos v)$

$\underline{x}_{uu} = R(-\cos u \cos v, -\sin u \cos v, 0)$, $\underline{x}_{uv} = R(\sin u \sin v, -\cos u \sin v, 0)$,

$\underline{x}_{vv} = R(-\cos u \cos v, -\sin u \cos v, -\sin v)$, $N = (\cos u \cos v, \sin u \cos v, \sin v)$

$E = \langle \underline{x}_u, \underline{x}_u \rangle = R^2 \cos^2 v$, $F = \langle \underline{x}_u, \underline{x}_v \rangle = 0$, $G = \langle \underline{x}_v, \underline{x}_v \rangle = R^2$

$e = \langle \underline{x}_{uu}, N \rangle = -R\cos^2 v$, $f = \langle \underline{x}_{uv}, N \rangle = 0$, $g = \langle \underline{x}_{vv}, N \rangle = -R$

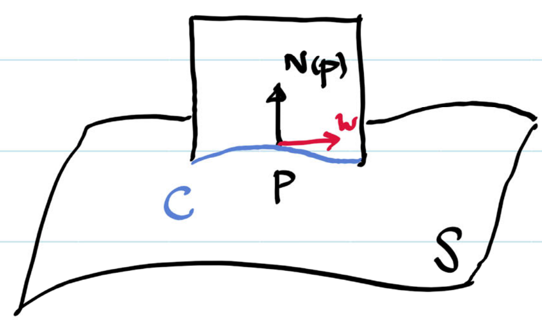
Computing $-\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{pmatrix}$ gives

$K = \frac{1}{R^2}$, $H = -\frac{1}{R}$ on the image of $\underline{x}$. Check this holds at every $p \in S$!

Indeed, $dN_p(w) = \frac{1}{R} w$ $\forall w \in T_p(S)$ $\forall p \in S$, i.e. $dN_p = \frac{1}{R} \mathrm{Id}$ $\forall p \in S$.

Normal curvatures, Meusnier's theorem and Principal curvatures

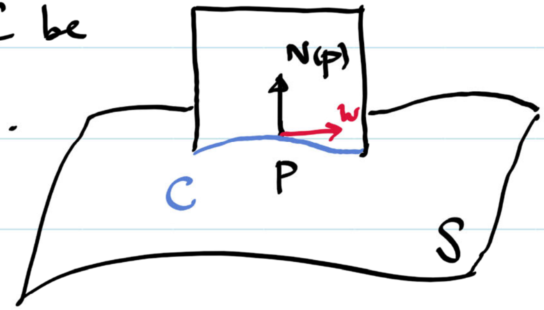
Let S be an oriented regular surface in $\mathbb{R}^3$ with a continuous choice of unit normal N on S, and let $p \in S$. If $w \in T_p(S)$ is a unit vector, the plane through p that contains both w and $N(p)$ intersects S on a curve C, that is called the normal section of S at p along w. We may parametrize this curve C using arc length parametrization, say $\alpha : (-\varepsilon, \varepsilon) \to C$ for some $\varepsilon > 0$ and $\alpha(0) = p$, and we may then compute the (unsigned) curvature of this space curve C at p, namely $k = |\alpha''(0)|$



(because $k = |\vec{T}'(0)|$ and $\vec{T}(s) = \alpha'(s)$ for arc length parametrized α). It does not matter which orientation we pick of α; k remains unchanged whether we parametrized C as $\alpha(s)$ or $\alpha(-s)$. We ask: what is the curvature k of this normal section C at the point p?

Theorem. The (unsigned) curvature of the normal section C to S at p along a unit vector $w \in T_p(S)$ is precisely $|\mathrm{II}_p(w)|$.

Proof. Let $p \in S$, $w \in T_p(S)$ with unit length, and C be the normal section to S at p along w.



Let $\alpha : I \to C$ be a parametrization of C by arc length with $\alpha(0) = p$ over some open interval $I \ni 0$, say with $\alpha'(0) = w$.

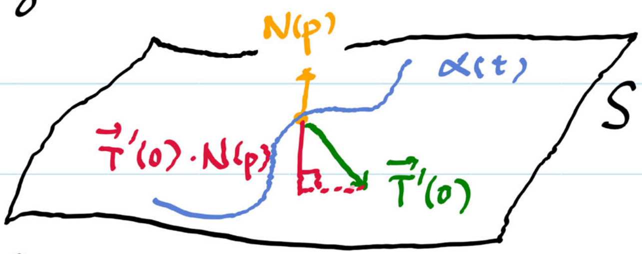
We need to compute $k := |\alpha''(0)|$.

But $\alpha''(0)$ is contained in the plane through p that contains w and $N(p)$ (because the curve C is), and $\alpha''(0)$ is orthogonal to $\alpha'(0) = w$ (because one can differentiate $\alpha'(s) \cdot \alpha'(s) = 1$).

Thus $\alpha''(0)$ is a multiple of $N(p)$. It follows that $k = |\alpha''(0)| = |\alpha''(0) \cdot N(\alpha(0))|$. But since $\alpha'(s) \cdot N(\alpha(s)) = 0 \ \forall s$, differentiation yields $\alpha''(0) \cdot N(\alpha(0)) = -\alpha'(0) \cdot dN_{\alpha(0)}(\alpha'(0)) = \mathrm{II}_p(w)$, the last equality holding because $\alpha(0) = p$ and $\alpha'(0) = w$. Done!

Remark: This shows $\vec{T}'(0) \cdot N(p) = \mathrm{II}_p(w)$ if $w \in T_p(S)$ is a unit vector.

Definition. Let S be any oriented regular surface with unit normal N. Let $\alpha(t)$ be any regular curve on S with $\alpha(0) = p \in S$. The normal curvature of α at p is then defined to be $k_n := \dfrac{\vec{T}'(0) \cdot N(p)}{|\alpha'(0)|}$ where $\vec{T}(t) = \dfrac{\alpha'(t)}{|\alpha'(t)|}$ is the unit tangent to $\alpha(t)$.



Theorem. (Meusnier) If $w = \alpha'(0) \in T_p(S)$ in the above definition, then the normal curvature of α at p is precisely $\dfrac{II_p(w)}{I_p(w)}$ where I_p, II_p are the first and second fundamental forms of S at p.

Proof. $\dfrac{\vec{T}'(0) \cdot N(p)}{|\alpha'(0)|} = \dfrac{1}{|w|} \dfrac{d}{dt}\left(\dfrac{\alpha'(t)}{|\alpha'(t)|}\right) \cdot N(\alpha(t)) \Big|_{t=0} = \dfrac{1}{|w|^2} \alpha''(t) \cdot N(\alpha(t)) \Big|_{t=0}$

Differentiate $N \cdot \alpha' = 0$, same as previous proof

$= -\dfrac{1}{|w|^2} \langle \alpha'(0), dN_p(\alpha'(0)) \rangle = \dfrac{II_p(w)}{I_p(w)}$ ▨

In particular, if $w \in T_p(S)$ with $|w| = 1$, then $II_p(w) =$ normal curvature of any regular curve $\alpha(t)$ on S with $\alpha(0) = p$ and $\alpha'(0) = w$!

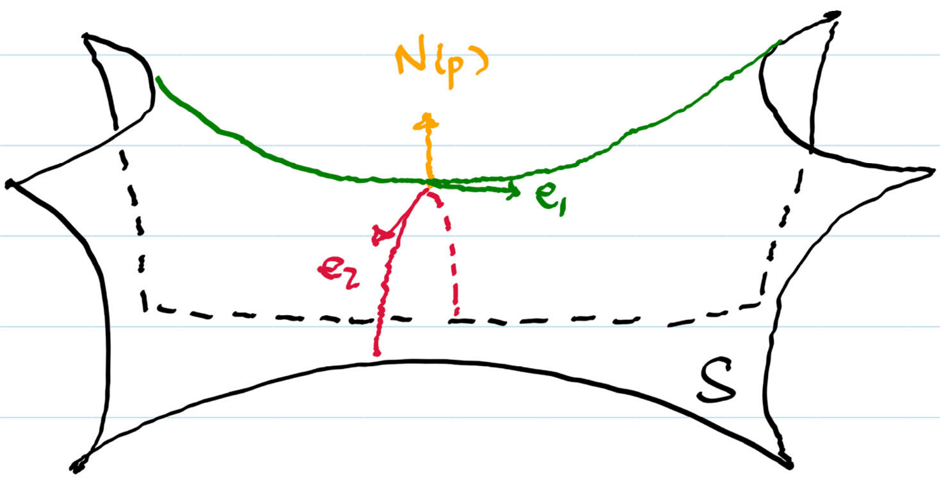
As a result, if S is an oriented regular surface, $p \in S$, and $w \in T_p(S)$ with $|w|=1$, we sometimes call $\mathrm{II}_p(w)$ the normal curvature of S at p along direction w, without reference to any particular regular curve $\alpha(t)$ on S.

Next we fix p on an oriented regular surface S and look at how the normal curvature $\mathrm{II}_p(w)$ varies as w varies over the set of unit vectors in $T_p(S)$. Recall there exists an orthonormal basis $\{e_1, e_2\}$ of $T_p(S)$, and real numbers $k_1, k_2 \in \mathbb{R}$, such that $-dN_p(e_1) = k_1 e_1$ and $-dN_p(e_2) = k_2 e_2$. In other words, e_1, e_2 are eigenvectors of $-dN_p$ with eigenvalues k_1, k_2. The eigenvalues k_1 and k_2 of $-dN_p$ are called the principal curvatures to S at p, and the directions in which e_1, e_2 point are called the principal directions at p. They can be computed using our earlier matrix representation $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ of dN_p (e.g. $k_1, k_2 =$ eigenvalues of $-\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, and eigenvectors to $-\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ can be used to express the principal directions in terms of the basis of $T_p(S)$ used to identify $-dN_p$ with $-\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.)

If w is a unit vector in $T_p(S)$, say $w = (\cos\theta)e_1 + (\sin\theta)e_2$ for some $\theta \in \mathbb{R}$, then $\mathrm{II}_p(w) = -\langle dN_p(w), w\rangle_p = k_1\cos^2\theta + k_2\sin^2\theta$, so if further $k_1 \geq k_2$, then

$$k_2 \leq \mathrm{II}_p(w) \leq k_1 .$$

This shows that at $p \in S$, as w varies over the set of unit vectors in $T_p(S)$, the normal curvature at p along direction w is the biggest (resp. smallest), when $w =$ the eigenvector of $-dN_p$ that corresponds to the bigger (resp. smaller) principal curvature k_1 (resp. k_2) at p.



$\{e_1, e_2\}$ orthonormal at p.

$\mathrm{II}_p(e_2) \leq \mathrm{II}_p(w) \leq \mathrm{II}_p(e_1)$

$\forall w \in T_p(S)$ with

$\langle w, w\rangle_p = 1$.

Example. Let S be the surface $\{(x,y,z)\in\mathbb{R}^3 : z = xy\}$, oriented by the unit normal that has a positive third component. Find the principal curvatures and principal directions at $(0,0,0)\in S$.

Solution. Parametrize S by $\mathbf{x}(u,v) = (u,v,uv)$ for $(u,v)\in\mathbb{R}^2$. Then

$\mathbf{x}_u = (1,0,v)$, $\mathbf{x}_u = (0,1,u)$, $\mathbf{x}_{uu} = \mathbf{x}_{vv} = (0,0,0)$, $\mathbf{x}_{uv} = (0,0,1)$,

$N = \dfrac{(-v,-u,1)}{\sqrt{1+u^2+v^2}}$. Hence $E = 1+v^2$, $F = uv$, $G = 1+u^2$, $e = g = 0$, $f = \dfrac{1}{\sqrt{1+u^2+v^2}}$.

Hence with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ of T_pS, $-dN_p$ is represented by

$$-\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\begin{pmatrix} 0 & \frac{1}{\sqrt{1+u^2+v^2}} \\ \frac{1}{\sqrt{1+u^2+v^2}} & 0 \end{pmatrix}\begin{pmatrix} 1+v^2 & uv \\ uv & 1+u^2 \end{pmatrix}^{-1} = -\frac{1}{(1+u^2+v^2)^{3/2}}\begin{pmatrix} -uv & 1+v^2 \\ 1+u^2 & -uv \end{pmatrix}.$$

At $p = \mathbf{x}(0,0)$ this matrix reduces to $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. We find its eigenvalues by solving $\det\begin{pmatrix} -\lambda & -1 \\ -1 & -\lambda \end{pmatrix} = 0$, i.e. $\lambda^2 - 1 = 0$, so the principal curvatures at $(0,0,0)$ are 1 and -1. The corresponding eigenvectors of the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively, so the corresponding principal directions are $1\cdot\mathbf{x}_u + (-1)\mathbf{x}_v = (1,-1,0)$ and $1\cdot\mathbf{x}_u + 1\cdot\mathbf{x}_v = (1,1,0)$.

Example. Let S be the surface $\{(x,y,z)\in\mathbb{R}^3 : z = x^2+2xy\}$, and $p=(\sqrt{2},\sqrt{2},6)\in S$. Find the principal curvatures of S at p.

Solution. Parametrize S by $\underline{x}(u,v)=(u,v,u^2+2uv)$. Then at $p=\underline{x}(\sqrt{2},\sqrt{2})$, with respect to the basis $\{\underline{x}_u,\underline{x}_v\}$ of $T_p(S)$, $-dN_p$ is represented by

$$-\begin{pmatrix} \frac{2}{\sqrt{41}} & \frac{2}{\sqrt{41}} \\ \frac{2}{\sqrt{41}} & 0 \end{pmatrix}\begin{pmatrix} 33 & 16 \\ 16 & 9 \end{pmatrix}^{-1} = \frac{2}{\sqrt{41}^3}\begin{pmatrix} 7 & -17 \\ -9 & 16 \end{pmatrix},$$ with eigenvalues $\frac{23\pm 3\sqrt{77}}{\sqrt{41}^3}$

So the principal curvatures at p are $\frac{23+3\sqrt{77}}{\sqrt{41}^3}$ and $\frac{23-3\sqrt{77}}{\sqrt{41}^3}$.

Example. Let S be an oriented regular surface, $p\in S$, and suppose the Gaussian and mean curvatures of S at p are -8 and 1 respectively. Find the principal curvatures of S at p.

Solution. The principal curvatures k_1 and k_2 satisfies $k_1k_2=-8$ and $\frac{k_1+k_2}{2}=1$. So they are roots of $\lambda^2-2\lambda-8=0$, and principal curvatures at p are 4 and -2.