A moving frame along a regular surface

Along a regular curve in $\mathbb{R}^2$ we had a Frenet frame $\{\vec{T}, \vec{N}\}$ of $\mathbb{R}^2$ and Frenet's formula describes how they vary along the curve:
we had $\begin{cases} \vec{T}' = kN \\ \vec{N}' = -kT \end{cases}$ if the curve is arc length parametrized,

where k is the curvature of the curve.

Let's do something similar on an oriented regular surface S in $\mathbb{R}^3$.
Let $\underline{x}: U \to \underline{x}(U) \subseteq S$ be a local parametrization defined on some open set $U \subseteq \mathbb{R}^2$. Then $\{\underline{x}_u, \underline{x}_v, N\}$ form a moving frame of $\mathbb{R}^3$ along S, where N is the unit normal to S given by its orientation.

Question: Can we study derivatives of $\underline{x}_u, \underline{x}_v, N$ along S?

2 possible partial derivatives (with respect to u and v) $\rightsquigarrow \begin{cases} \underline{x}_{uu}, \underline{x}_{uv}, \\ \underline{x}_{vu}, \underline{x}_{vv}, \\ N_u, N_v \end{cases}$

Will that give us some relation to the curvature of S?

Goal: At any $p \in \mathbf{x}(U)$, express $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, $\mathbf{x}_{vu}$, $\mathbf{x}_{vv}$, N_u, N_v in terms of $\mathbf{x}_u, \mathbf{x}_v, N$ (Possible because $\{\mathbf{x}_u, \mathbf{x}_v, N\}$ at p form a basis of $\mathbb{R}^3$).

This is easy given what we had done:

① $\begin{cases} N_u = dN_p(\mathbf{x}_u) \\ N_v = dN_p(\mathbf{x}_v) \end{cases} \Rightarrow \begin{cases} N_u = a_{11}\mathbf{x}_u + a_{12}\mathbf{x}_v \\ N_v = a_{21}\mathbf{x}_u + a_{22}\mathbf{x}_v \end{cases}$ where $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$

is the matrix representation of the differential of the Gauss map at p in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ of $T_p(S)$; here E, F, G and e, f, g are the first and second fundamental forms on S at p.

② $\mathbf{x}_{uu} = \nabla_{\mathbf{x}_u}\mathbf{x}_u$ + some multiple of N. Taking dot product with N, we get $\mathbf{x}_{uu} \cdot N$ = (some multiple of N) $\cdot N$ $\Rightarrow \mathbf{x}_{uu} = \nabla_{\mathbf{x}_u}\mathbf{x}_u + eN$.

This gives $\mathbf{x}_{uu} = \Gamma_{11}^1 \mathbf{x}_u + \Gamma_{11}^2 \mathbf{x}_v + eN$

Similarly, $\mathbf{x}_{uv} = \Gamma_{12}^1 \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_v + fN = \mathbf{x}_{vu}$

$\mathbf{x}_{vv} = \Gamma_{22}^1 \mathbf{x}_u + \Gamma_{22}^2 \mathbf{x}_v + gN$.

To get to curvatures we need to differentiate once more.

Note that $(\mathbf{x}_{uu})_v = (\mathbf{x}_{uv})_u$, so ① $\langle (\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u, \mathbf{x}_v \rangle = 0$ and ② $\langle (\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u, N \rangle = 0$

① implies $R^1_{211} F + R^2_{211} G = eg - f^2$ where
$$\begin{cases} R^1_{211} = \partial_v \Gamma^1_{11} - \partial_u \Gamma^1_{12} + \Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22} - \Gamma^1_{12}\Gamma^1_{11} - \Gamma^2_{12}\Gamma^1_{21} \\ R^2_{211} = \partial_v \Gamma^2_{11} - \partial_u \Gamma^2_{12} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{21} \end{cases}$$

(Gauss equation)

Indeed, $(\mathbf{x}_{uu})_v = (\Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + eN)_v = (\partial_v\Gamma^1_{11})\mathbf{x}_u + \Gamma^1_{11}\mathbf{x}_{uv} + (\partial_v\Gamma^2_{11})\mathbf{x}_v + \Gamma^2_{11}\mathbf{x}_{vv} + e_v N + eN_v$

$= (\partial_v\Gamma^1_{11} + \Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22})\mathbf{x}_u + (\partial_v\Gamma^2_{11} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22})\mathbf{x}_v + eN_v + (e_v + \Gamma^1_{11}f + \Gamma^2_{11}g)N$

$(\mathbf{x}_{uv})_u = (\Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + fN)_u = (\partial_u\Gamma^1_{12})\mathbf{x}_u + \Gamma^1_{12}\mathbf{x}_{uu} + (\partial_u\Gamma^2_{12})\mathbf{x}_v + \Gamma^2_{12}\mathbf{x}_{vu} + f_u N + fN_u$

$= (\partial_u\Gamma^1_{12} + \Gamma^1_{12}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{21})\mathbf{x}_u + (\partial_u\Gamma^2_{12} + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{21})\mathbf{x}_v + fN_u + (f_u + \Gamma^1_{12}e + \Gamma^2_{12}f)N$

So $(\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u = R^1_{211}\mathbf{x}_u + R^2_{211}\mathbf{x}_v + eN_v - fN_u + (e_v + \Gamma^1_{11}f + \Gamma^2_{11}g - f_u - \Gamma^1_{12}e - \Gamma^2_{12}f)N$. Since $\langle \mathbf{x}_u, \mathbf{x}_v\rangle = F$, $\langle \mathbf{x}_v, \mathbf{x}_v\rangle = G$, $\langle N, \mathbf{x}_v\rangle = 0$, $\langle N_v, \mathbf{x}_v\rangle = -g$ and $\langle N_u, \mathbf{x}_v\rangle = -f$, ① implies $R^1_{211}F + R^2_{211}G = eg - f^2$.

Similarly, since $\langle \mathbf{x}_u, N\rangle = \langle \mathbf{x}_v, N\rangle = \langle N_u, N\rangle = \langle N_v, N\rangle = 0$, from ② we obtain that

(Codazzi equations)
$$\begin{cases} e_v - \Gamma^1_{12}e - \Gamma^2_{12}f = f_u - \Gamma^1_{11}f - \Gamma^2_{11}g \\ f_v - \Gamma^1_{22}e - \Gamma^2_{22}f = g_u - \Gamma^1_{12}f - \Gamma^2_{12}g \end{cases}$$
; we also have, from ③ $\langle (\mathbf{x}_{uv})_v - (\mathbf{x}_{vv})_u, N\rangle = 0$, that

c.f. Fundamental theorem of surfaces

Remarkably the Gauss and Codazzi equations are the only compatibility conditions that three functions e, f, g has to satisfy, for them to be the second fundamental forms of a regular surface in $\mathbb{R}^3$ with prescribed first fundamental form E, F, G.

Gaussian curvature revisited

Now define $R_{2112} := R^1_{211} F + R^2_{211} G$ $(= \langle R^1_{211} x_u + R^2_{211} x_v, x_v \rangle)$, so that the Gauss equation may simply be written as $R_{2112} = eg - f^2$. Then the Gaussian curvature K on S can be calculated as follows.

Recall $K = \dfrac{eg-f^2}{EG-F^2}$. Hence Gauss equation implies $K = \dfrac{R_{2112}}{EG-F^2}$.

This is a remarkable formula, because
$$\begin{cases} R^1_{211} = \partial_v \Gamma^1_{11} - \partial_u \Gamma^1_{12} + \Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22} - \Gamma^1_{12}\Gamma^1_{11} - \Gamma^2_{12}\Gamma^1_{21} \\ R^2_{211} = \partial_v \Gamma^2_{11} - \partial_u \Gamma^2_{12} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{21} \end{cases}$$
depend only on the Christoffel symbols Γ^k_{ij} on S, which in turn depend only on the first fundamental form E, F, G on S!

<u>Theorem Egregium (Gauss)</u>. The Gaussian curvature K of a regular surface S in $\mathbb{R}^3$ depends only on the first fundamental form on S.

So the Gaussian curvature is invariant under (local) isometries.

Definition. Let S_1, S_2 be regular surfaces in $\mathbb{R}^3$.

① S_1 and S_2 are said to be isometric, if there exists a diffeomorphism $f: S_1 \to S_2$ such that $\langle df_p(v_1), df_p(v_2) \rangle = \langle v_1, v_2 \rangle$ for every $p \in S_1$ and every $v_1, v_2 \in T_p(S_1)$.

② If W is an open subset of S_1 and $f: W \to f(W) \subseteq S_2$ is an isometry between W and $f(W)$, then f is called a local isometry.

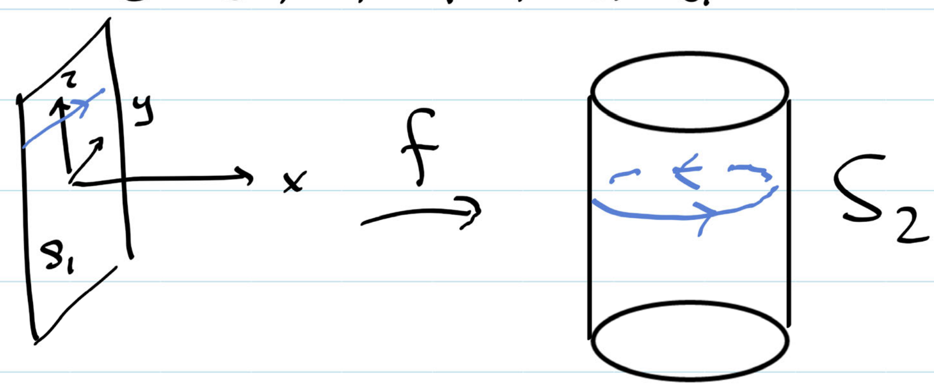
Proposition. If there exists an open set $U \subseteq \mathbb{R}^2$ and local parametrizations $x: U \to x(U) \subseteq S_1$, $\tilde{x}: U \to \tilde{x}(U) \subseteq S_2$ such that the coefficients of the first fundamental forms satisfy

$$E = \tilde{E}, \quad F = \tilde{F}, \quad G = \tilde{G}$$

(where $E = \langle x_u, x_u \rangle$, $F = \langle x_u, x_v \rangle$, $G = \langle x_v, x_v \rangle$, $\tilde{E} = \langle \tilde{x}_u, \tilde{x}_u \rangle$, $\tilde{F} = \langle \tilde{x}_u, \tilde{x}_v \rangle$ and $\tilde{G} = \langle \tilde{x}_v, \tilde{x}_v \rangle$), then $\tilde{x} \circ x^{-1}: x(U) \to \tilde{x}(U)$ is an isometry between $x(U)$ and $\tilde{x}(U)$, hence a local isometry from S_1 into S_2.

Example 1. Let $S_1 = \{(x,y,z) \in \mathbb{R}^3 : x = 0\}$ be the y-z plane and $S_2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1\}$ be a cylinder. If $W = \{(0,y,z) \in \mathbb{R}^3 : |y| < \pi\} \subseteq S_1$ and $f : W \to S_2$ is defined by $f(0,y,z) = (\cos y, \sin y, z)$, then f is an isometry between W and $f(W)$; indeed, if $\underline{x}(u,v) = (0,u,v) \in S_1$, $\tilde{\underline{x}}(u,v) = (\cos u, \sin u, v) \in S_2$, both defined for $(u,v) \in \mathcal{U} := (-\pi,\pi) \times \mathbb{R}$, then it is easily checked $E = \langle \underline{x}_u, \underline{x}_u \rangle = 1$, $F = \langle \underline{x}_u, \underline{x}_v \rangle = 0$, $G = \langle \underline{x}_v, \underline{x}_v \rangle = 1$, while $\tilde{E} = \langle \tilde{\underline{x}}_u, \tilde{\underline{x}}_u \rangle = 1$ (since $\tilde{\underline{x}}_u = (-\sin u, \cos u, 0)$),

$\tilde{F} = \langle \tilde{\underline{x}}_u, \tilde{\underline{x}}_v \rangle = 0$, $\tilde{G} = \langle \tilde{\underline{x}}_v, \tilde{\underline{x}}_v \rangle = 1$ so we do have $E = \tilde{E}$, $F = \tilde{F}$, $G = \tilde{G}$.



Example 2. Let $\underline{x}(u,v) = (\cosh v \cos u, \cosh v \sin u, v)$, $(u,v) \in (-\pi, \pi) \times (0, \infty)$ be a parametrization of (part of) the catenoid S_1, and $\underline{y}(u,v) = (v\cos u, v \sin u, u)$, $(u,v) \in (-\pi, \pi) \times (0,\infty)$ be a parametrization of the helicoid S_2.

Then they are locally isometric, because

$$\begin{cases} \underline{x}_u = (-\cosh v \sin u, & \cosh v \cos u, \ 0) \\ \underline{x}_v = (\sinh v \cos u, & \sinh v \sin u, \ 1) \end{cases}$$

$\Rightarrow E = \langle \underline{x}_u, \underline{x}_u \rangle = \cosh^2 v$, $F = \langle \underline{x}_u, \underline{x}_v \rangle = 0$, $G = \langle \underline{x}_v, \underline{x}_v \rangle = \cosh^2 v$,

while if $\tilde{\underline{x}}(u,v) := \underline{y}(u, \sinh v) = (\sinh v \cos u, \sinh v \sin u, u)$, then

$$\begin{cases} \tilde{\underline{x}}_u = (-\sinh v \sin u, \ \sinh v \cos u, \ 1) \\ \tilde{\underline{x}}_v = (\cosh v \cos u, \ \cosh v \sin u, \ 0) \end{cases}$$

$\Rightarrow \tilde{E} = \langle \tilde{\underline{x}}_u, \tilde{\underline{x}}_u \rangle = \cosh^2 v$, $\tilde{F} = \langle \tilde{\underline{x}}_u, \tilde{\underline{x}}_v \rangle = 0$, $\tilde{G} = \langle \tilde{\underline{x}}_v, \tilde{\underline{x}}_v \rangle = \cosh^2 v$.

Hence $E = \tilde{E}$, $F = \tilde{F}$, $G = \tilde{G}$, and $\tilde{\underline{x}} \circ \underline{x}^{-1} : \underline{x}(U) \to \tilde{\underline{x}}(U)$ is an isometry if $U = (-\pi, \pi) \times (0, \infty)$.

(Check out youtube videos for a deformation of a catenoid into a helicoid).

If S_1, S_2 are regular surfaces in $\mathbb{R}^3$ and $\underline{x}: \mathcal{U} \to \underline{x}(\mathcal{U}) \subseteq S_1$, $\tilde{\underline{x}}: \mathcal{U} \to \tilde{\underline{x}}(\mathcal{U}) \subseteq S_2$ are local parametrizations defined on some common open set $\mathcal{U} \subseteq \mathbb{R}^2$ with $E = \tilde{E}$, $F = \tilde{F}$, $G = \tilde{G}$, then the Christoffel symbols Γ_{ij}^k on $\underline{x}(\mathcal{U}) \subseteq S_1$ and $\tilde{\Gamma}_{ij}^k$ on $\tilde{\underline{x}}(\mathcal{U}) \subseteq S_2$ are the same. Hence $R_{2112} = \tilde{R}_{2112}$, and the Gaussian curvature K of S_1 at $\underline{x}(u_0, v_0)$ is the same as the Gaussian curvature $\tilde{K}$ of S_2 at $\tilde{\underline{x}}(u_0, v_0)$, for any $(u_0, v_0) \in \mathcal{U}$. This is because $K = \dfrac{R_{2112}}{EG - F^2} = \dfrac{\tilde{R}_{2112}}{\tilde{E}\tilde{G} - \tilde{F}^2} = \tilde{K}$.

Example. The Gaussian curvature of the cylinder $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ is identically zero (because the plane has zero Gaussian curvature and the cylinder is locally isometric to the plane).

Remark: Only the Gaussian curvature is preserved under local isometries; the mean curvature is not. For example, the plane $\{(x,y,z) \in \mathbb{R}^3 : x = 0\}$ has zero mean curvature, but the cylinder $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2=1\}$ has mean curvature equal to $\frac{0+1}{2} = \frac{1}{2}$ (because the principal curvatures of this cylinder is 0 and 1 respectively at every point), even though the plane and the cylinder are locally isometric.

We say the Gaussian curvature is an intrinsic property of a regular surface, meaning that it can be determined by measurements (of the first fundamental form) within the surface; it is independent of how the surface is embedded in $\mathbb{R}^3$ (contrary to say the mean curvature)

To summarize, the Gaussian curvature $= \dfrac{eg-f^2}{EG-F^2}$ can be calculated using the first fundamental form E, F, G, because $eg-f^2 = R_{2112}$ where

$$R_{2112} = R^1_{211}F + R^2_{211}G \quad \text{and} \quad \begin{cases} R^1_{211} = \partial_v\Gamma^1_{11} - \partial_u\Gamma^1_{12} + \Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22} - \Gamma^1_{12}\Gamma^1_{11} - \Gamma^2_{12}\Gamma^1_{21} \\ R^2_{211} = \partial_v\Gamma^2_{11} - \partial_u\Gamma^2_{12} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{21}. \end{cases}$$

Question. How can one remember such a complicated formula?

Theorem. Let $x : U \to x(U) \subseteq S$ be a local parametrization of a regular surface S. Then $R_{2112} = \langle (\nabla_{x_v}\nabla_{x_u} - \nabla_{x_u}\nabla_{x_v})x_u, x_v \rangle$.

Proof. $\nabla_{x_u}x_u = \Gamma^1_{11}x_u + \Gamma^2_{11}x_v$, $\quad \nabla_{x_v}x_u = \Gamma^1_{12}x_u + \Gamma^2_{12}x_v$, so

$$\nabla_{x_v}\nabla_{x_u}x_u = (\partial_v\Gamma^1_{11})x_u + \Gamma^1_{11}\nabla_{x_v}x_u + (\partial_v\Gamma^2_{11})x_v + \Gamma^2_{11}\nabla_{x_v}x_v$$
$$= (\partial_v\Gamma^1_{11})x_u + \Gamma^1_{11}\Gamma^1_{12}x_u + \Gamma^1_{11}\Gamma^2_{12}x_v + (\partial_v\Gamma^2_{11})x_v + \Gamma^2_{11}\Gamma^1_{22}x_u + \Gamma^2_{11}\Gamma^2_{22}x_v,$$

$$\nabla_{x_u}\nabla_{x_v}x_u = (\partial_u\Gamma^1_{12})x_u + \Gamma^1_{12}\nabla_{x_u}x_u + (\partial_u\Gamma^2_{12})x_v + \Gamma^2_{12}\nabla_{x_u}x_v$$
$$= (\partial_u\Gamma^1_{12})x_u + \Gamma^1_{12}\Gamma^1_{11}x_u + \Gamma^1_{12}\Gamma^2_{11}x_v + (\partial_u\Gamma^2_{12})x_v + \Gamma^2_{12}\Gamma^1_{21}x_u + \Gamma^2_{12}\Gamma^2_{21}x_v,$$

so $(\nabla_{x_v}\nabla_{x_u} - \nabla_{x_u}\nabla_{x_v})x_u = R^1_{211}x_u + R^2_{211}x_v \Rightarrow \langle(\nabla_{x_v}\nabla_{x_u} - \nabla_{x_u}\nabla_{x_v})x_u, x_v\rangle = R_{2112}$.

Notation. For $i,j,k,l \in \{1,2\}$, define $R_{ijkl} := \langle (\nabla_{x_i}\nabla_{x_j} - \nabla_{x_j}\nabla_{x_i})x_k, x_l \rangle$ where $\begin{cases} x_1 := x_u \\ x_2 := x_v. \end{cases}$

Fact. In 2 dimensions, $R_{2112} = R_{1221} = -R_{1212} = -R_{2121} = eg-f^2$, all other $R_{ijkl} = 0$.

Example. Calculate the Gaussian curvature of the sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$ using the Christoffel symbols.

Solution. Parametrize S^2 by $\underline{x}(u,v) = (\cos u \cos v, \cos u \sin v, \sin u)$

We saw $\underline{x}_u = (-\sin u \cos v, -\sin u \sin v, \cos u)$

$$\underline{x}_v = (-\cos u \sin v, \cos u \cos v, 0)$$

So $E = 1$, $F = 0$, $G = \cos^2 u$.

From $E_u = E_v = 0$, $F_u = F_v = 0$, $G_u = -2\sin u \cos u$, $G_v = 0$,

we have $\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{22}^2 = 0$, $\Gamma_{12}^2 = -\tan u$, $\Gamma_{22}^1 = \sin u \cos u$.

Hence $R_{2112} = R_{211}^1 \cancelto{0}{F} + R_{211}^2 G = R_{211}^2 \cos^2 u$, but

$$R_{211}^2 = \cancelto{0}{\partial_2 \Gamma_{11}^2} - \partial_1 \Gamma_{12}^2 + \cancelto{0}{\Gamma_{11}^1} \Gamma_{12}^2 + \cancelto{0}{\Gamma_{11}^2} \Gamma_{22}^2 - \cancelto{0}{\Gamma_{12}^1} \Gamma_{11}^2 - \Gamma_{12}^2 \Gamma_{21}^2$$

$$= -\partial_u(-\tan u) - (-\tan u)^2 = \sec^2 u - \tan^2 u = 1 .$$

It follows that the Gaussian curvature at $\underline{x}(u,v)$ is

$K = \dfrac{R_{2112}}{EG-F^2} = \dfrac{1 \cdot \cos^2 u}{\cos^2 u} = 1$. By rotational symmetry, the Gaussian curvature is 1 at every point on S^2.

Alternatively, from $\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{22}^2 = 0$, $\Gamma_{12}^2 = -\tan u$, $\Gamma_{22}^1 = \sin u \cos u$, we compute $\nabla_{\underline{x}_u} \underline{x}_u = 0$, $\nabla_{\underline{x}_v} \underline{x}_u = -\tan u\, \underline{x}_v$,

$\nabla_{\underline{x}_v} \nabla_{\underline{x}_u} \underline{x}_u - \nabla_{\underline{x}_u} \nabla_{\underline{x}_v} \underline{x}_u = -\nabla_{\underline{x}_u}(-\tan u\, \underline{x}_v) = \frac{\partial}{\partial u}(\tan u)\, \underline{x}_v + \tan u\, \nabla_{\underline{x}_u} \underline{x}_v$

$= \sec^2 u\, \underline{x}_v - \tan^2 u\, \underline{x}_v = \underline{x}_v$, so $R_{2112} = \langle \underline{x}_v, \underline{x}_v \rangle = \cos^2 u$, and the Gaussian curvature at $\underline{x}(u,v)$ is $K = \frac{R_{2112}}{EG-F^2} = \frac{1\cdot\cos^2 u}{\cos^2 u} = 1$.

Example. Calculate the Gaussian curvature of the paraboloid $S = \{(x,y,z) \in \mathbb{R}^3 : z = x^2+y^2\}$ using Christoffel symbols.

Solution. Parametrize S by $\underline{x}(u,v) = (u, v, u^2+v^2)$. Then $\underline{x}_u = (1,0,2u)$, $\underline{x}_v = (0,1,2v)$, so $E = 1+4u^2$, $F = 4uv$, $G = 1+4v^2$, and $EG-F^2 = 1+4u^2+4v^2$.

Hence
$$\begin{cases} \Gamma_{11}^1 E + \Gamma_{11}^2 F = \underline{x}_{uu}\cdot\underline{x}_u = \frac{1}{2}E_u = 4u \\ \Gamma_{11}^1 F + \Gamma_{11}^2 G = \underline{x}_{uu}\cdot\underline{x}_v = F_u - \frac{1}{2}E_v = 4v \end{cases}$$

$$\Rightarrow \begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} 4u \\ 4v \end{pmatrix} = \frac{1}{1+4u^2+4v^2}\begin{pmatrix} 1+4v^2 & -4uv \\ -4uv & 1+4u^2 \end{pmatrix}\begin{pmatrix} 4u \\ 4v \end{pmatrix}$$

$$\Rightarrow \Gamma_{11}^1 = \frac{4u}{1+4u^2+4v^2}, \quad \Gamma_{11}^2 = \frac{4v}{1+4u^2+4v^2}.$$

Similarly $\begin{cases} \Gamma_{12}^1 E + \Gamma_{12}^2 F = \mathbf{x}_{uv}\cdot \mathbf{x}_u = \frac{1}{2}E_v = 0 \\ \Gamma_{12}^1 F + \Gamma_{12}^2 G = \mathbf{x}_{uv}\cdot \mathbf{x}_v = \frac{1}{2}G_u = 0 \end{cases} \Rightarrow \Gamma_{12}^1 = \Gamma_{12}^2 = 0,$

and $\begin{cases} \Gamma_{22}^1 E + \Gamma_{22}^2 F = \mathbf{x}_{vv}\cdot \mathbf{x}_u = F_v - \frac{1}{2}G_u = 4u \\ \Gamma_{22}^1 F + \Gamma_{22}^2 G = \mathbf{x}_{vv}\cdot \mathbf{x}_v = \frac{1}{2}G_v = 4v \end{cases} \Rightarrow \begin{cases} \Gamma_{22}^1 = \frac{4u}{1+4u^2+4v^2} \\ \Gamma_{22}^2 = \frac{4v}{1+4u^2+4v^2}. \end{cases}$

So $(\nabla_{\mathbf{x}_v}\nabla_{\mathbf{x}_u} - \nabla_{\mathbf{x}_u}\nabla_{\mathbf{x}_v})\mathbf{x}_u = \nabla_{\mathbf{x}_v}\left(\frac{4u}{1+4u^2+4v^2}\mathbf{x}_u + \frac{4v}{1+4u^2+4v^2}\mathbf{x}_v\right) - \nabla_{\mathbf{x}_u}(0\,\mathbf{x}_u + 0\,\mathbf{x}_v)$

$$= \frac{-32uv}{(1+4u^2+4v^2)^2}\mathbf{x}_u + \frac{4u}{1+4u^2+4v^2}\cancelto{0}{\nabla_{\mathbf{x}_v}\mathbf{x}_u} + \frac{4(1+4u^2-4v^2)}{(1+4u^2+4v^2)^2}\mathbf{x}_v + \frac{4v}{1+4u^2+4v^2}\nabla_{\mathbf{x}_v}\mathbf{x}_v$$

$$= \frac{-32uv+16uv}{(1+4u^2+4v^2)^2}\mathbf{x}_u + \frac{4(1+4u^2-4v^2)+16v^2}{(1+4u^2+4v^2)^2}\mathbf{x}_v$$

$$= \frac{-16uv}{(1+4u^2+4v^2)^2}\mathbf{x}_u + \frac{4(1+4u^2)}{(1+4u^2+4v^2)^2}\mathbf{x}_v$$

As a result, $R_{2112} = -\frac{16uv}{(1+4u^2+4v^2)^2}\cdot 4uv + \frac{4(1+4u^2)}{(1+4u^2+4v^2)^2}(1+4v^2) = \frac{4}{1+4u^2+4v^2}.$

It follows that the Gaussian curvature at $\mathbf{x}(u,v)$ is

$$K = \frac{R_{2112}}{EG-F^2} = \frac{4}{(1+4u^2+4v^2)^2}.$$

(Remark: This is not the fastest way to compute Gaussian curvature ...)