Covariant differentiation along curves

Let S be a regular surface in $\mathbb{R}^3$, and $\gamma : I \to S$ be a C^∞ curve on S. A vector field w along γ is an association to each $t \in I$ some $w(t) \in T_{\gamma(t)}(S)$. In local coordinates, we may write

always tangent to S!

$w(t) = w_1(t)\, \mathbf{x}_u + w_2(t)\, \mathbf{x}_v$ (where $\mathbf{x}_u, \mathbf{x}_v$ are evaluated at $\gamma(t)$).

w is said to be C^∞ if $w_1(t)$ and $w_2(t)$ are C^∞ functions of $t \in I$. (This is well-defined independent of the choice of local parametrization $\mathbf{x}$.)

Example. Fix $\theta \in (-\pi, \pi)$. Then $\gamma(t) = (\cos t \cos\theta, \cos t \sin\theta, \sin t)$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, parametrizes a longitude on the sphere $\mathbb{S}^2$.

If $w(t) = \gamma'(t) \in T_{\gamma(t)}(\mathbb{S}^2)$ $\forall t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then $w(t)$ defines a C^∞ vector field along γ.

(Note that in this case, w is not defined away from γ; there is no obvious way to define w at a point on $\mathbb{S}^2 \setminus \gamma(I)$.)

Our goal is to covariant differentiate a C^∞ vector field $w(t)$ along $\gamma(t)$. We knew how to do so if $w(t) = \tilde{w}(\gamma(t))$ for some C^∞ vector field $\tilde{w}$ defined in an open set on S containing $\gamma(I)$: in that case, we may define the covariant derivative of $w(t)$ along $\gamma(t)$ to be $\nabla_{\gamma'(t)}\tilde{w}$; if $\tilde{w}(\mathbf{x}(u,v)) = \tilde{w}_1(\mathbf{x}(u,v))\,\mathbf{x}_u(u,v) + \tilde{w}_2(\mathbf{x}(u,v))\,\mathbf{x}_v(u,v)$ and $\gamma(t) = \mathbf{x}(u(t),v(t))$ so that $\gamma'(t) = u'(t)\mathbf{x}_u + v'(t)\mathbf{x}_v$ at $\gamma(t)$, then this covariant derivative is given by $\nabla_{u'(t)\mathbf{x}_u + v'(t)\mathbf{x}_v}(\tilde{w}_1\mathbf{x}_u + \tilde{w}_2\mathbf{x}_v)$

$$= \frac{d}{dt}\left(\tilde{w}_1(\gamma(t))\right)\mathbf{x}_u + \frac{d}{dt}(\tilde{w}_2(\gamma(t))\,\mathbf{x}_v + \tilde{w}_1(\gamma(t))\,u'(t)\,\nabla_{\mathbf{x}_u}\mathbf{x}_u + \tilde{w}_1(\gamma(t))\,v'(t)\,\nabla_{\mathbf{x}_v}\mathbf{x}_u$$

$$+ \tilde{w}_2(\gamma(t))\,u'(t)\,\nabla_{\mathbf{x}_u}\mathbf{x}_v + \tilde{w}_2(\gamma(t))\,v'(t)\,\nabla_{\mathbf{x}_v}\mathbf{x}_v$$

later write $u_1(t)$ for $u(t)$, $u_2(t)$ for $v(t)$

depends only on $w(t)$ and $\gamma(t)$!

$$= \left(\frac{dw_1}{dt} + u'(t)w_1(t)\Gamma^1_{11} + v'(t)w_1(t)\Gamma^1_{12} + u'(t)w_2(t)\Gamma^1_{21} + v'(t)w_2(t)\Gamma^1_{22}\right)\mathbf{x}_u$$

$$+\left(\frac{dw_2}{dt} + u'(t)w_1(t)\Gamma^2_{11} + v'(t)w_1(t)\Gamma^2_{12} + u'(t)w_2(t)\Gamma^2_{21} + v'(t)w_2(t)\Gamma^2_{22}\right)\mathbf{x}_v$$

where $w_1(t) = \tilde{w}_1(\gamma(t))$ and $w_2(t) = \tilde{w}_2(\gamma(t))$; here Γ^k_{ij} are evaluated at $\gamma(t)$ and $\mathbf{x}_u, \mathbf{x}_v$ are evaluated at $\mathbf{x}^{-1}(\gamma(t))$.

Since the restriction of $\nabla_{\gamma'(t)} \tilde{w}$ to $\gamma(t)$ depends only on $w(t)$ (and $\gamma(t)$) but not the extension $\tilde{w}$ of $w(t)$, we may introduce the following definition:

Definition (Covariant differentiation of a C^∞ vector field along γ)
Let $\underline{x} : U \to \underline{x}(U) \subseteq S$ be a local parametrization of a regular surface S in $\mathbb{R}^3$. Let $\gamma(t) = \underline{x}(u_1(t), u_2(t))$ be a C^∞ curve on S, and $w(t)$ be a C^∞ vector field along γ. Write $w(t) = w_1(t)\,\underline{x}_1 + w_2(t)\,\underline{x}_2$ (evaluated at $\underline{x}^{-1}(\gamma(t))$). Then define the covariant derivative of $w(t)$ along $\gamma(t)$ to be

$$\frac{Dw}{dt} := \sum_{k=1}^{2} \left(\frac{dw_k}{dt} + \sum_{i,j=1}^{2} \frac{du_i}{dt}\, w_j(t)\, \Gamma_{ij}^{k}(\gamma(t)) \right) \underline{x}_k$$

$\underline{x}_1 := \underline{x}_u$
$\underline{x}_2 := \underline{x}_v$
in our previous notations

or more succinctly

$$\frac{Dw}{dt} = \sum_{k=1}^{2} \left(\frac{dw_k}{dt} + \sum_{i,j=1}^{2} \frac{du_i}{dt}\, w_j \Gamma_{ij}^{k} \right) \underline{x}_k$$

Note $\frac{Dw}{dt} \in T_{\gamma(t)}(S)$; indeed $\frac{Dw}{dt}$ is the orthogonal projection of the coordinatewise derivative $\frac{dw}{dt}$ to $T_{\gamma(t)}(S)$. ($\frac{Dw}{dt}$: covariant derivative of $w(t)$ along $\gamma(t)$; $\frac{dw}{dt}$: componentwise derivative of $w(t)$)

How do we remember this formula for $\frac{Dw}{dt}$?

In fact, we only need to remember the following properties of $\frac{D}{dt}$:

① $$\frac{D}{dt}\left(w_1(t)\mathbf{x}_u + w_2(t)\mathbf{x}_v\right) = \frac{dw_1}{dt}\mathbf{x}_u + w_1(t)\frac{D\mathbf{x}_u}{dt} + \frac{dw_2}{dt}\mathbf{x}_v + w_2(t)\frac{D\mathbf{x}_v}{dt}.$$

② $$\frac{D\mathbf{x}_u}{dt} = \nabla_{\gamma'(t)}\mathbf{x}_u \quad \text{and} \quad \frac{D\mathbf{x}_v}{dt} = \nabla_{\gamma'(t)}\mathbf{x}_v.$$

{which is the case if $\gamma(t) = \mathbf{x}(u(t), v(t))$}

In other words, if $\gamma'(t) = u'(t)\mathbf{x}_u + v'(t)\mathbf{x}_v$, then

$$\begin{cases} \dfrac{D\mathbf{x}_u}{dt} = u'(t)\nabla_{\mathbf{x}_u}\mathbf{x}_u + v'(t)\nabla_{\mathbf{x}_v}\mathbf{x}_u = u'(t)\left(\Gamma_{11}^1\mathbf{x}_u + \Gamma_{11}^2\mathbf{x}_v\right) + v'(t)\left(\Gamma_{12}^1\mathbf{x}_u + \Gamma_{12}^2\mathbf{x}_v\right) \\ \dfrac{D\mathbf{x}_v}{dt} = u'(t)\nabla_{\mathbf{x}_u}\mathbf{x}_v + v'(t)\nabla_{\mathbf{x}_v}\mathbf{x}_v = u'(t)\left(\Gamma_{12}^1\mathbf{x}_u + \Gamma_{12}^2\mathbf{x}_v\right) + v'(t)\left(\Gamma_{22}^1\mathbf{x}_u + \Gamma_{22}^2\mathbf{x}_v\right). \end{cases}$$

(Remember to differentiate $\mathbf{x}_u$ and $\mathbf{x}_v$ when we covariant differentiate!)

(Also remember, from now on we will write $\mathbf{x}_1 := \mathbf{x}_u$, $\mathbf{x}_2 := \mathbf{x}_v$ if it is convenient to do so).

Example. Let $\underset{\sim}{x}(\phi,\theta) = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$ be a local parametrization of S^2. Fix $\theta \in (-\pi,\pi)$. Let $\gamma(t) = \underset{\sim}{x}(t,\theta)$ for $t \in (-\frac{\pi}{2},\frac{\pi}{2})$. Let $w(t) = t^2 \underset{\sim}{x}_2 \in T_{\gamma(t)}(S^2)$. Find $\frac{Dw}{dt}$ along $\gamma(t)$.

Solution. Since $\underset{\sim}{x}_1 = \underset{\sim}{x}_\phi = (-\sin\phi\cos\theta, -\sin\phi\sin\theta, \cos\phi)$,

$\underset{\sim}{x}_2 = \underset{\sim}{x}_\theta = (-\cos\phi\sin\theta, \cos\phi\cos\theta, 0)$,

we have $E = 1$, $F = 0$, $G = \cos^2\phi$, and $\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{22}^2 = 0$, $\Gamma_{12}^2 = -\tan\phi$, $\Gamma_{22}^1 = \sin\phi\cos\phi$. Also, $\gamma'(t) = \underset{\sim}{x}_1$, $w(t) = t^2\underset{\sim}{x}_2$.

$$\text{So } \frac{Dw}{dt} = \frac{d}{dt}(t^2)\underset{\sim}{x}_2 + t^2\frac{D\underset{\sim}{x}_2}{dt} = 2t\,\underset{\sim}{x}_2 + t^2\,\nabla_{\underset{\sim}{x}_1}\underset{\sim}{x}_2$$

$$= 2t\,\underset{\sim}{x}_2 + t^2\cancelto{0}{\Gamma_{12}^1}\underset{\sim}{x}_1 + t^2\Gamma_{12}^2\underset{\sim}{x}_2 = 2t\,\underset{\sim}{x}_2 - t^2\tan(t)\,\underset{\sim}{x}_2$$

$$= (2t - t^2\tan(t))(-\cos t\cos\theta, \cos t\sin\theta, 0) \in T_{\gamma(t)}(S^2).$$

Proposition. If $w(t), \tilde{w}(t)$ are (tangent) vector field along a C^∞ curve γ on a regular surface S, then $\frac{d}{dt}\langle w(t), \tilde{w}(t)\rangle = \langle \frac{Dw}{dt}, \tilde{w}\rangle + \langle w, \frac{D\tilde{w}}{dt}\rangle$.

Proof.

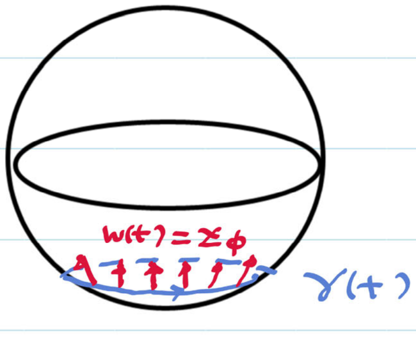
$$\frac{d}{dt}\langle w(t), \tilde{w}(t)\rangle \underset{\substack{\uparrow\\ \text{product}\\ \text{rule}}}{=} \langle \frac{dw}{dt}, \tilde{w}\rangle + \langle w, \frac{d\tilde{w}}{dt}\rangle \underset{\substack{\uparrow\\ \frac{dw}{dt} - \frac{Dw}{dt} \perp T_{\gamma(t)}(S) \text{ and same for } \tilde{w}.}}{=} \langle \frac{Dw}{dt}, \tilde{w}\rangle + \langle w, \frac{D\tilde{w}}{dt}\rangle.$$

Definition. A C^∞ vector field w along γ is said to be parallel, if $\frac{Dw}{dt} = 0$ along γ. Informally we think of w as "constant" along γ.

Example. Let $\underline{x}(\phi, \theta) = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$ for $\theta \in (-\pi, \pi)$, $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ so that it parametrizes the unit sphere (minus a longitude). Fix $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, consider the latitude $\gamma(t) = \underline{x}(\phi, t)$. Let $w(t) = \underline{x}_\phi$ along $\gamma(t)$. Is $w(t)$ parallel along γ?

Solution. Since $\gamma'(t) = \underline{x}_2$ and $w(t) = \underline{x}_1$, $\frac{Dw}{dt}$ is simply $\nabla_{\underline{x}_2} \underline{x}_1$ $= \Gamma_{12}^1 \underline{x}_1 + \Gamma_{12}^2 \underline{x}_2$. Since $\Gamma_{12}^1 = 0$, $\Gamma_{12}^2 = -\tan\phi$, we have $\frac{Dw}{dt} = -\tan\phi\, \underline{x}_\theta$, which is the zero vector if and only if $\tan\phi = 0$, i.e. if and only if $\phi = 0$. So the vector field $w(t)$ is parallel along γ if $\phi = 0$ (i.e. if γ is the equator), and not parallel along γ if $\phi \neq 0$ (i.e. if γ is a latitude which is not the equator).

Remark. Here is a picture describing the previous example:



We said $w(t) = X_\phi$ is not parallel (or "constant") along γ if γ is a latitude other than the equator.

This may sound strange because $w(t) = 1 \cdot X_\phi$ and the coefficient 1 is constant along γ.

But this is not so strange, because we do not have any reason to think of X_ϕ as "not changing" when we move from one tangent space $T_{\gamma(t_1)}(S^2)$ to another $T_{\gamma(t_2)}(S^2)$.

Here's a famous physics experiment that gives us a more accurate intuition about parallel vector fields. Think S^2 = Earth. A pendulum is swung in the north-south direction at a city on $\gamma(0)$. As Earth rotates, the pendulum's position is at $\gamma(t)$ at time t, and if the pendulum is left to swing "freely", then after a while the pendulum does not swing along a north-south plane, unless the city is on the equator! More about this soon.

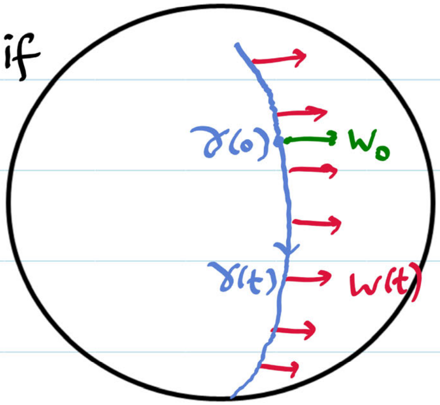
Parallel transport of a vector along a curve

Theorem. Let I be an open interval containing 0 and let $\gamma : I \to S$ be a C^∞ curve on a regular surface S in $\mathbb{R}^3$. Then for any $w_0 \in T_{\gamma(0)}(S)$, there exists a unique vector field $w(t)$ along γ, such that w is parallel along γ, and such that $w(0) = w_0$.

We say such $w(t)$ is the parallel transport of w_0 along γ.
Indeed, if $\gamma'(t) = \frac{du_1}{dt}\mathbf{x}_1 + \frac{du_2}{dt}\mathbf{x}_2$ and $w_0 = w_{01}\mathbf{x}_1 + w_{02}\mathbf{x}_2 \in T_{\gamma(0)}(S)$, then $w(t) = w_1(t)\mathbf{x}_1 + w_2(t)\mathbf{x}_2$ is parallel transport of w_0, if

$$\begin{cases} \frac{Dw}{dt} = 0 \\ w(0) = w_0 \end{cases}, \text{i.e.} \begin{cases} \frac{dw_k}{dt} = -\sum_{i,j=1}^{2} \frac{du_i}{dt} w_j(t)\,\Gamma_{ij}^{k}(\gamma(t)) \\ w_k(0) = w_{0k} \end{cases} \text{for } k=1,2.$$

Theory of ordinary differential equations show that a solution $(w_1(t), w_2(t))$ exists and is unique on I.



Example. Let $S = \{(x,y,z) \in \mathbb{R}^3 : z=0\}$ be the x-y plane in $\mathbb{R}^3$.

Let γ be any C^∞ curve on S, and $w_0 = (a,b,0) \in T_{\gamma(0)}(S)$ for some $a, b \in \mathbb{R}$. What is the parallel transport of w_0 along γ?

Solution. Let's parametrize the x-y plane by $\underline{x}(u,v) = (u,v,0)$ $\forall (u,v) \in \mathbb{R}^2$.

Then $\underline{x}_u = (1,0,0)$ and $\underline{x}_v = (0,1,0)$, so $w_0 = a\underline{x}_u + b\underline{x}_v \in T_{\gamma(0)}(S)$.

Let $w(t) = w_1(t)\underline{x}_u + w_2(t)\underline{x}_v$ be the parallel transport of w_0.

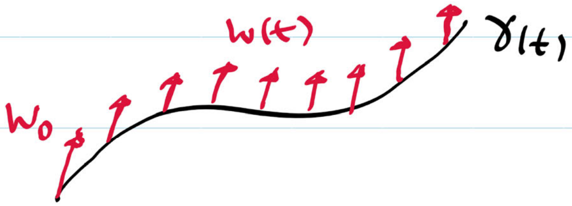
Then $\begin{cases} \frac{Dw}{dt} = 0 & \text{— ①} \\ w(0) = w_0 & \text{— ②} \end{cases}$ From ② we have $\begin{cases} w_1(0) = a \\ w_2(0) = b. \end{cases}$

From ① we have $w_1'(t)\underline{x}_u + w_1(t)\nabla_{\gamma'(t)}\underline{x}_u + w_2'(t)\underline{x}_v + w_2(t)\nabla_{\gamma'(t)}\underline{x}_v = 0$ which simplifies to $w_1'(t)\underline{x}_u + w_2'(t)\underline{x}_v = 0$ since $\Gamma_{ij}^k = 0$ for all $i,j,k \in \{1,2\}$ (which follows since E, F, G are all constants)

So $w_1'(t) = w_2'(t) = 0$, meaning that $w_1(t), w_2(t)$ are both constants.

Hence $w_1(t) = a$, $w_2(t) = b$, and

$w(t) = a\underline{x}_u + b\underline{x}_v = (a,b,0)$ $\forall t$.



Example. Let $\underline{x}(\phi,\theta) = (\cos\phi\cos\theta,\ \cos\phi\sin\theta,\ \sin\phi)$ be a local parametrization of the sphere S^2 (minus a longitude). Fix $\phi\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $\omega>0$. Let $\gamma(t)=\underline{x}(\phi,\omega t)$ and let $w_0=\underline{x}_\phi(\phi,0)\in T_{\gamma(0)}(S^2)$.

Find the parallel transport of w_0 along γ.

Solution. Let $w(t) = w_1(t)\underline{x}_\phi + w_2(t)\underline{x}_\theta$ be the parallel transport of w_0 along γ. Then since $\gamma'(t) = \omega\underline{x}_\theta = \omega\underline{x}_2$, we have

$$\frac{Dw}{dt} = \frac{dw_1}{dt}\underline{x}_1 + w_1\nabla_{\omega\underline{x}_2}\underline{x}_1 + \frac{dw_2}{dt}\underline{x}_2 + w_2\nabla_{\omega\underline{x}_2}\underline{x}_2$$

$$0 = \frac{Dw}{dt} = \frac{dw_1}{dt}\underline{x}_1 + w_1(t)\,\omega\big(\cancelto{0}{\Gamma^1_{12}}\underline{x}_1 + \underbrace{\Gamma^2_{12}}_{-\tan\phi}\underline{x}_2\big) + \frac{dw_2}{dt}\underline{x}_2 + w_2(t)\,\omega\big(\underbrace{\Gamma^1_{22}}_{\sin\phi\cos\phi}\underline{x}_1 + \cancelto{0}{\Gamma^2_{22}}\underline{x}_2\big)$$

i.e. $$\begin{cases} \dfrac{dw_1}{dt} = -\omega\sin\phi\cos\phi\, w_2(t) \\ \dfrac{dw_2}{dt} = \omega\tan\phi\, w_1(t) \end{cases}$$ with initial conditions $w_1(0)=1$, $w_2(0)=0$.

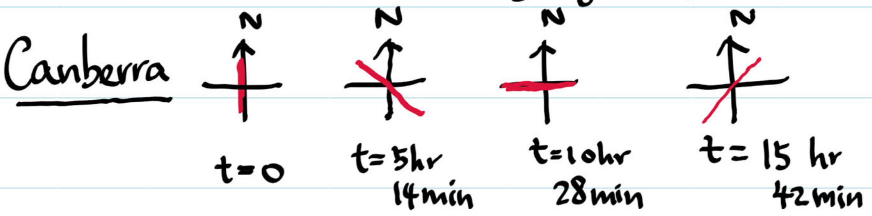
It follows that $\frac{d^2w_1}{dt^2} = -\omega\sin\phi\cos\phi\frac{dw_2}{dt} = -\omega\sin\phi\cos\phi\,\omega\tan\phi\, w_1 = -\omega^2\sin^2\phi\, w_1$, so $w_1(t) = A\cos(\omega t\sin\phi) + B\sin(\omega t\sin\phi)$ for some constants A, B, and the initial conditions $w_1(0)=1$, $w_1'(0) = -\omega\sin\phi\cos\phi\, w_2(0) = 0$ gives $A=1$, $B=0$. Hence $w_1(t) = \cos(\omega t\sin\phi)$, and then

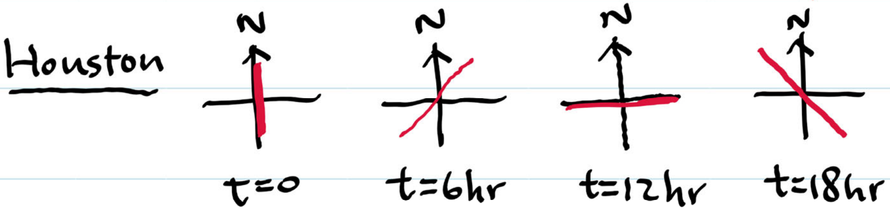
$$w_2(t) = \int_0^t \omega\tan\phi\, w_1(t)\,dt = \int_0^t \omega\tan\phi\cos(\omega t\sin\phi)\,dt = \frac{1}{\cos\phi}\sin(\omega t\sin\phi)$$

i.e. $w(t) = \cos(\omega t\sin\phi)\,\underline{x}_\phi + \sin(\omega t\sin\phi)\,\frac{\underline{x}_\theta}{\cos\phi}$.

Physical interpretation (Foucault's pendulum)

The previous example shows that if $w_0 = \underline{x}_\phi$ at $\underline{x}(\phi, 0)$, then $w(t) = \cos(\omega t \sin\phi)\,\underline{x}_\phi + \sin(\omega t \sin\phi)\,\frac{\underline{x}_\theta}{\cos\phi}$ is the parallel transport of w_0 along the latitude $\gamma(t) := \underline{x}(\phi, \omega t)$. This is indeed what happens to a pendulum that initially swings in the north-south (i.e. $\underline{x}_\phi$) direction, if $\omega := \frac{2\pi}{24\,\text{hours}}$ is the angular velocity of the rotation of Earth: the direction of swing at time t would make an angle $\omega t \sin\phi$ with the north (because $\underline{x}_\phi$ and $\frac{\underline{x}_\theta}{\cos\phi}$ are unit vectors pointing north and east respectively), meaning its direction of swing will slowly change, and it takes time $\frac{\alpha}{|\omega \sin\phi|}$ for the direction of swing to turn an angle α. e.g. $\phi = -35^\circ$ at Canberra, so if a pendulum is released to swing freely in the north-south plane at Canberra, then the direction of swing gradually become NW-SE then west-east; and $\phi = 30^\circ$ at Houston, so the direction of swing goes from N-S to NE-SW to E-W. Below red line = direction of swing.





Geometric interpretation

Let S be a regular surface in $\mathbb{R}^3$ and $\gamma : I \to S$ be a C^∞ curve on S. The parallel transport along γ provides, for each $t \in I$, a linear map $P_t : T_{\gamma(0)}(S) \to T_{\gamma(t)}(S)$. This map turns out to be an isometry, meaning that $\langle w_1, w_2 \rangle_{\gamma(0)} = \langle P_t w_1, P_t w_2 \rangle_{\gamma(t)}$ $\forall\, w_1, w_2 \in T_{\gamma(0)}(S)$. This can be seen by observing that $P_t w_1$ and $P_t w_2$ are parallel vector fields along γ (so $\frac{D}{dt}(P_t w_1) \equiv 0$ and $\frac{D}{dt}(P_t w_2) \equiv 0$), and hence $\frac{d}{dt}\langle P_t w_1, P_t w_2 \rangle = \langle \cancelto{0}{\frac{D}{dt}(P_t w_1)}, P_t w_2 \rangle + \langle P_t w_1, \cancelto{0}{\frac{D}{dt}(P_t w_2)} \rangle = 0$, meaning that $\langle P_t w_1, P_t w_2 \rangle = \langle P_0 w_1, P_0 w_2 \rangle = \langle w_1, w_2 \rangle$ $\forall\, t \in I$. This map $P_t : T_{\gamma(0)}(S) \to T_{\gamma(t)}(S)$ is the "correct" way of identifying $T_{\gamma(0)}(S)$ with $T_{\gamma(t)}(S)$ (and hence between any two tangent spaces of S along γ): if $w_1 \in T_{\gamma(t_1)}(S)$ and $w_2 \in T_{\gamma(t_2)}(S)$ then we should identify w_1 with w_2 if $w_2 = P_{t_2} P_{t_1}^{-1} w_1$, i.e. if w_2 can be obtained from w_1 by parallel transport along γ (Necessarily w_2 and w_1 have the same length then, since each P_t is an isometry.)

To reiterate: Let $\gamma(t)$ be a C^∞ curve on a regular surface S in $\mathbb{R}^3$.

① If $w(t)$ and $\tilde{w}(t)$ are parallel vector fields along $\gamma(t)$, must $\langle w(t), \tilde{w}(t)\rangle$ be a constant independent of t?

Yes. Indeed, $\frac{d}{dt}\langle w(t), \tilde{w}(t)\rangle = \langle \cancelto{0}{\frac{Dw}{dt}}, \tilde{w}(t)\rangle + \langle w(t), \cancelto{0}{\frac{D\tilde{w}}{dt}}\rangle = 0$.

② If $w(t)$ is a parallel vector field along $\gamma(t)$, must $|w(t)|$ be a constant independent of t?

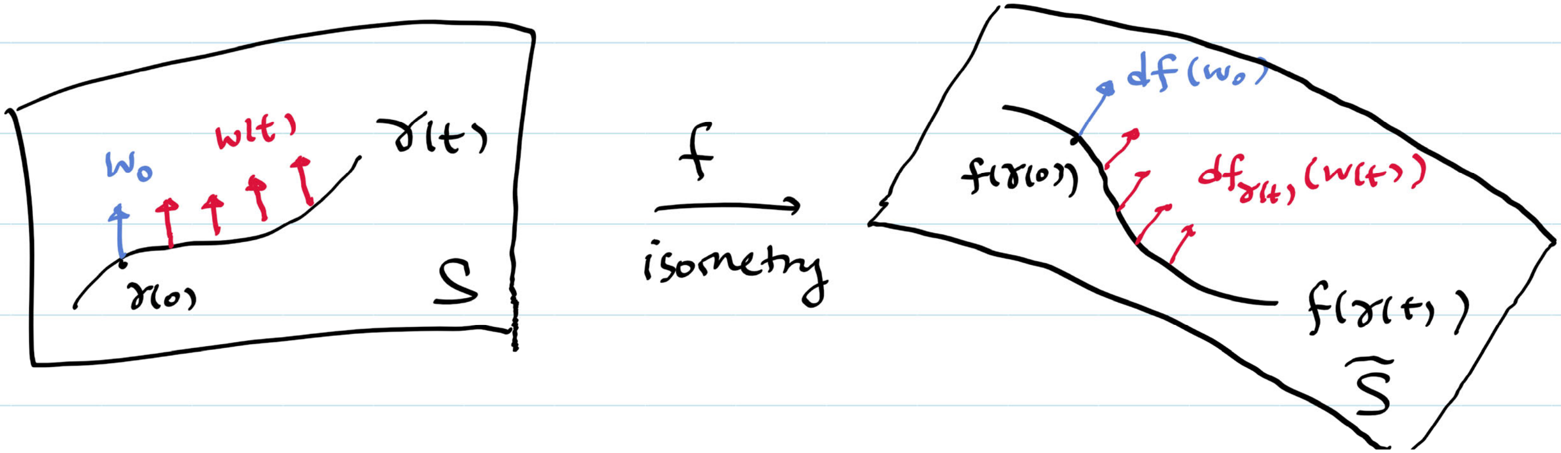
Yes. Indeed, $|w(t)|^2 = \langle w(t), w(t)\rangle$ is the dot product of two parallel vector fields along γ, and hence is constant by ①.

③ If $w(t), \tilde{w}(t)$ are parallel, non-zero vector fields along $\gamma(t)$, must the angle between $w(t)$ and $\tilde{w}(t)$ be constant independent of t?

Yes. The angle between $w(t)$ and $\tilde{w}(t)$ is $\arccos\left(\frac{\langle w(t), \tilde{w}(t)\rangle}{|w(t)|\,|\tilde{w}(t)|}\right)$ which is constant independent of t by ① and ②.

Further techniques in computing parallel transports

① If $S, \tilde{S}$ are isometric regular surfaces in $\mathbb{R}^3$, then as long as one knows how to parallel transport along a curve in S, one would know how to parallel transport along the corresponding curve on $\tilde{S}$. This is because the equations for parallel transport depend only on the Christoffel symbols, which depend only on the first fundamental form. More precisely, if $f: S \to \tilde{S}$ is an isometry, and $w(t)$ is the parallel transport of a vector $w_0 \in T_{\gamma(0)}(S)$ along a C^∞ curve γ on S, then $df_{\gamma(t)}(w(t))$ is the parallel transport of $df_{\gamma(0)}(w_0)$ along $f(\gamma)$ on $\tilde{S}$.



Example. For fixed $\phi \in (0, \pi)$, the map $\underline{x}(u,v) = (u\sin\phi\cos v, u\sin\phi\sin v, u\cos\phi)$ where $(u,v) \in \mathcal{U} := (0,\infty)\times(-\pi,\pi)$ parametrizes a cone (minus a ray) of aperture ϕ. For fixed $u_0 > 0$, the curve $\gamma(t) = \underline{x}(u_0,t)$ is a horizontal circle on the cone. If $w_0 = \underline{x}_u(u_0,0)$, find the parallel transport of w_0 along γ.

Solution. Idea: Cut open the cone and unfold it isometrically on a flat table. Let S be the sector in the x-y plane with angle $2\pi\sin\phi$, parametrized by $\tilde{\underline{x}}(u,v) = (u\sin(v\sin\phi), u\cos(v\sin\phi), 0)$ with $(u,v)\in\mathcal{U} = (0,\infty)\times(-\pi,\pi)$. The map $f(\tilde{\underline{x}}(u,v)) = \underline{x}(u,v)$ $\forall (u,v)\in\mathcal{U}$ defines an isometry from S to $\underline{x}(\mathcal{U}) \subseteq$ cone, $w_0 = df(\tilde{w}_0)$ if $\tilde{w}_0 = \tilde{\underline{x}}_u(u,0)$, and $f(\tilde{\gamma}(t)) = \gamma(t)$ if $\tilde{\gamma}(t) := \tilde{\underline{x}}(u_0,t)$. The parallel transport $\tilde{w}(t)$ of $\tilde{w}_0$ along $\tilde{\gamma}(t)$ is rotation of $\tilde{\underline{x}}_u(\gamma(t))$ by angle $-\frac{t}{\sin\phi}$, from which we may determine $w(t)$.

Check

$E=\tilde{E}$, $F=\tilde{F}$, $G=\tilde{G}$ where

$E=\langle \underline{x}_u, \underline{x}_u\rangle$

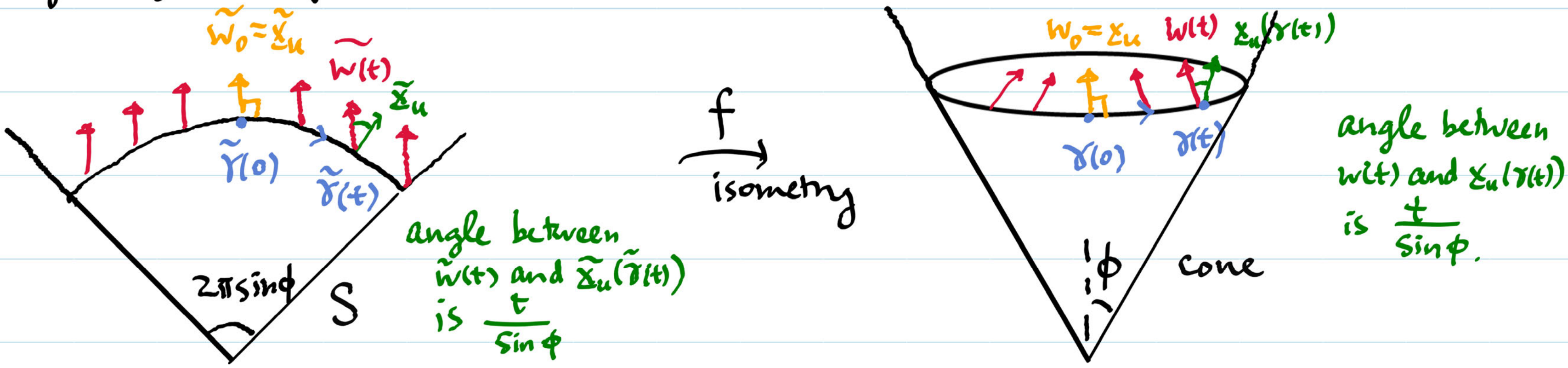
$F=\langle \underline{x}_u, \underline{x}_v\rangle$

$G=\langle \underline{x}_v, \underline{x}_v\rangle$,

$\tilde{E}=\langle \tilde{\underline{x}}_u, \tilde{\underline{x}}_u\rangle$

$\tilde{F}=\langle \tilde{\underline{x}}_u, \tilde{\underline{x}}_v\rangle$

$\tilde{G}=\langle \tilde{\underline{x}}_v, \tilde{\underline{x}}_v\rangle$.



angle between $\tilde{w}(t)$ and $\tilde{\underline{x}}_u(\tilde{\gamma}(t))$ is $\frac{t}{\sin\phi}$

angle between $w(t)$ and $\underline{x}_u(\gamma(t))$ is $\frac{t}{\sin\phi}$.

② If S and $\tilde{S}$ are regular surfaces, $\gamma(t)$ lies on both S and $\tilde{S}$, and $T_{\gamma(t)}S = T_{\gamma(t)}(\tilde{S})$, then parallel transport on S along γ coincides with parallel transport on $\tilde{S}$ along γ. Indeed, if $w(t) \in T_{\gamma(t)}(S) = T_{\gamma(t)}(\tilde{S})$ $\forall t$, then its covariant derivative along γ on S is the same as its covariant derivative along γ on $\tilde{S}$: both are the orthogonal projection of $\frac{dw}{dt}$ onto $T_{\gamma(t)}(S) = T_{\gamma(t)}(\tilde{S})$. Hence $w(t)$ is parallel on S if and only if $w(t)$ is parallel on $\tilde{S}$.

Example. Fix $\phi \in (-\frac{\pi}{2}, 0)$ and consider the latitude $\gamma(t) = (\cos\phi \cos t, \cos\phi \sin t, \sin\phi)$ on the unit sphere $\mathbb{S}^2$. $\gamma(t)$ also lies on a cone of aperture $|\phi|$ (consider the revolution of the following picture), and we knew how to parallel transport along a cone of aperture $|\phi|$. This gives us an alternative way to calculate the parallel transport along a latitude on the unit sphere.