Geodesics

Definition. A C^∞ parametrized curve $\gamma: I \to S$ on a regular surface S is called a geodesic on S if $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$ $\forall t \in I$ (i.e. if $\frac{d\gamma}{dt}$ is parallel along $\gamma(t)$).

In a local coordinate chart $\underline{x}: \mathcal{U} \to \underline{x}(\mathcal{U}) \subseteq S$, if $\gamma(t) = \underline{x}(u_1(t), u_2(t))$, then γ is a geodesic, if and only if $\frac{d^2u_k}{dt^2} + \sum_{i,j=1}^{2} \Gamma_{ij}^k(\gamma(t)) \frac{du_i}{dt}\frac{du_j}{dt} \equiv 0$ for $k=1,2$.

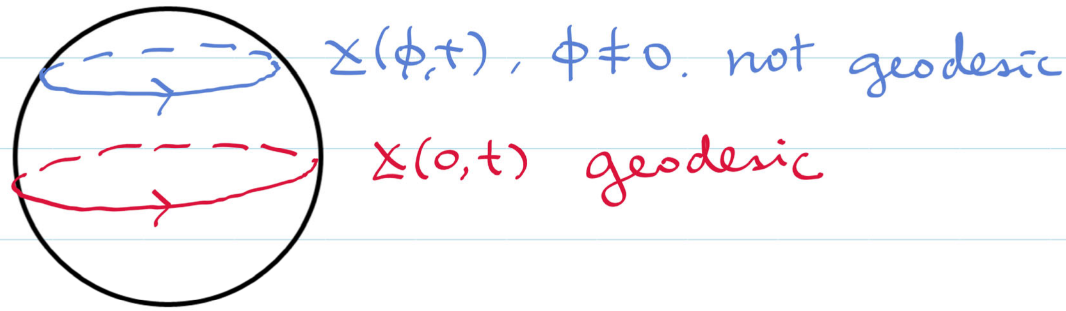
Example. Let $S = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$ be the x-y plane in $\mathbb{R}^3$.

Then $\gamma(t) = (u_1(t), u_2(t), 0)$ is a geodesic if and only if $\frac{d^2u_1}{dt} = 0$ and $\frac{d^2u_2}{dt} = 0$ (because all $\Gamma_{ij}^k = 0$), i.e.

if and only if $u_1(t) = a_1 t + b_1$, $u_2(t) = a_2 t + b_2$ for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$. This is equivalent to $\gamma(t) = (b_1, b_2, 0) + t(a_1, a_2, 0)$, i.e. γ is a straight line on S (with constant speeds). Hence geodesics are generalizations of "straight lines" to regular surfaces.

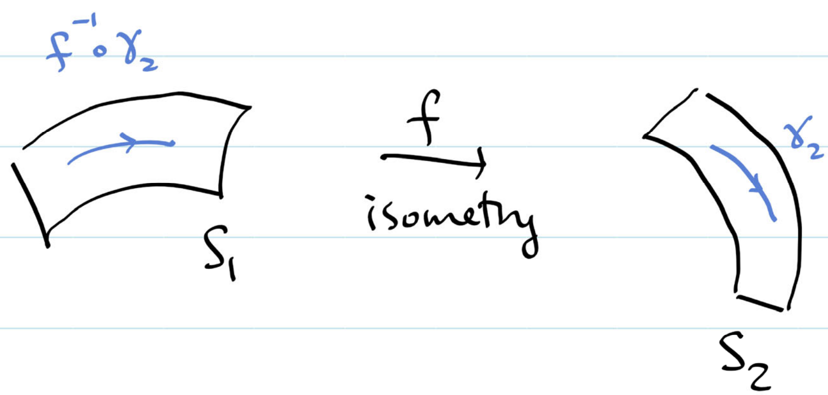
Example. Let $S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ be the unit sphere. Fix $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Let $\gamma(t) = (\cos\phi \cos t, \cos\phi \sin t, \sin\phi)$, $t \in (-\pi, \pi)$ be the latitude at angle ϕ. Show that if $\phi = 0$ (in which case γ parametrizes the equator at constant speed), then γ is a geodesic, and if $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ but $\phi \neq 0$, then γ is not a geodesic.

Solution. As before let $\underline{x}(u,v) = (\cos u \cos v, \cos u \sin v, \sin u)$ for $u \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $v \in (-\pi, \pi)$. Then $\gamma(t) = \underline{x}(\phi, t)$, so $\frac{d\gamma}{dt} = \underline{x}_v(\phi, t)$. We need to calculate $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right)$ $= \nabla_{\underline{x}_v} \underline{x}_v = \Gamma^1_{22}\, \underline{x}_u + \Gamma^2_{22}\, \underline{x}_v$ where we evaluate $\Gamma^1_{22}, \Gamma^2_{22}, \underline{x}_u, \underline{x}_v$ at $(u,v) = (\phi, t)$. But $\Gamma^1_{22} = \sin u \cos u$ and $\Gamma^2_{22} = 0$. So at $(u,v) = (\phi, t)$, we have $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \sin\phi \cos\phi\, \underline{x}_u(\phi, t)$, which is the zero vector if and only if $\phi = 0$.



The following proposition is often helpful in finding geodesics on regular surfaces.

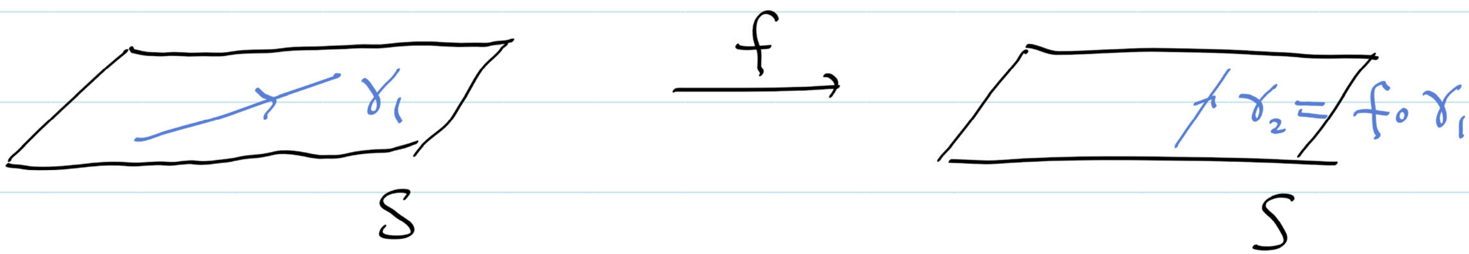
Proposition. Let S_1, S_2 be regular surfaces in $\mathbb{R}^3$ and $f: S_1 \to S_2$ be an isometry. Let $\gamma_2(t)$ be a C^∞ curve on S_2. Then $\gamma_2(t)$ is a geodesic on S_2, if and only if $f^{-1} \circ \gamma_2(t)$ is a geodesic on S_1.



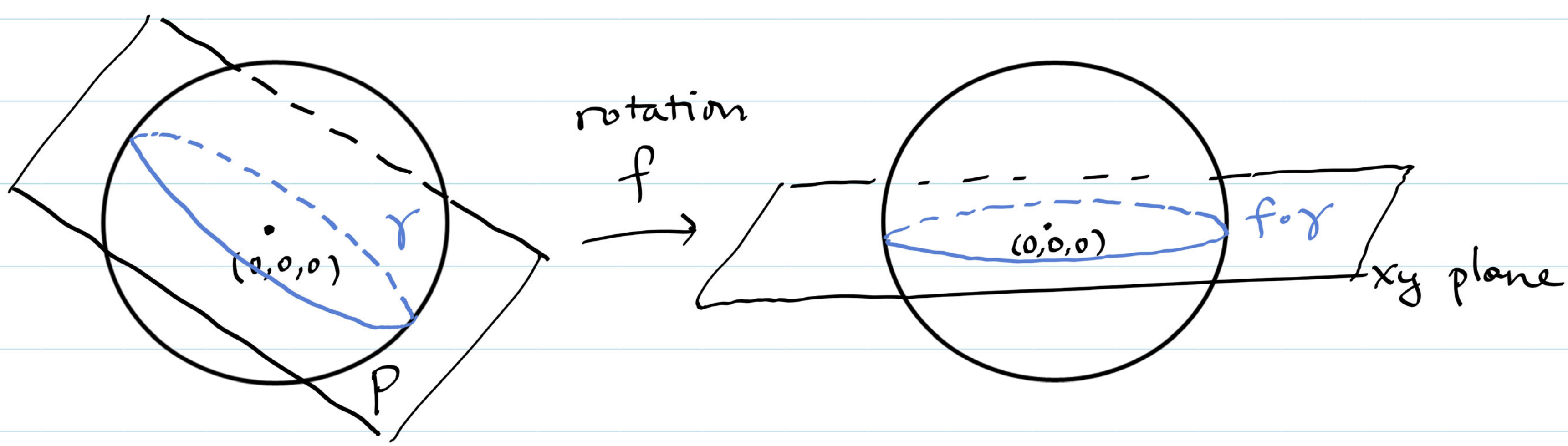
Proof. Let $\gamma_1(t) = f^{-1} \circ \gamma_2(t)$. Then $\frac{d\gamma_1}{dt} = df\left(\frac{d\gamma_2}{dt}\right)$. Since $f: S_1 \to S_2$ is an isometry, $\frac{d\gamma_1}{dt}$ is parallel along γ_1, if and only if $\frac{d\gamma_2}{dt}$ is parallel along γ_2.
(Alternatively, this follows by inspecting the geodesic equation, since the geodesic equation is determined by the Christoffel symbols, which are determined up to isometries).

Example. Let $S = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$ be the x–y plane. Let $f: S \to S$ be an isometry (i.e. $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M\begin{pmatrix} x \\ y \\ z \end{pmatrix} + b$ where M is a 3×3 matrix $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ being an orthogonal matrix, and $b = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix}$ is a vector in $\mathbb{R}^3$ with last entry 0).

Then f sends a straight line γ_1 on S, to another straight line γ_2 on S: if γ_1 passes through a point $p \in S$ and goes in direction v, then $\gamma_2 = f \circ \gamma_1$ is a straight line passing through $f(p)$ and goes in direction Mv. Hence f maps geodesics on S to geodesics on S.

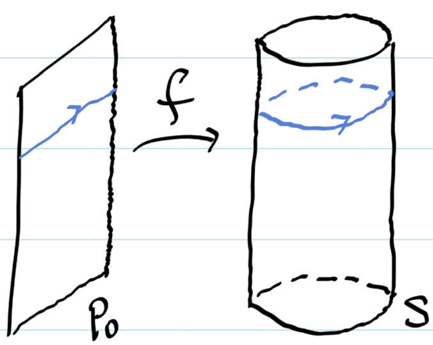


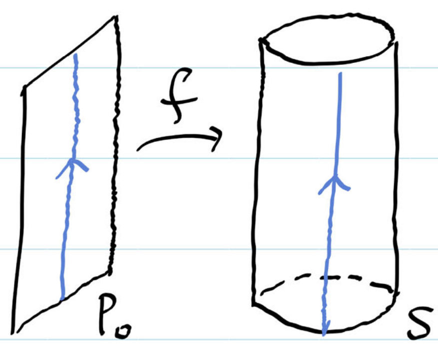
Example. If $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$ is the unit sphere and $\gamma(t)$ parametrizes a great circle on S^2 (i.e. the intersection of some plane P through $(0,0,0)$ with S^2) at unit speed, then $\gamma(t)$ is a geodesic on S^2. Indeed, given such a parametrized curve γ, there exists a rotation $f: \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(P) =$ the x-y plane. Such f restricts to an isometry $S^2 \to S^2$, and $f \circ \gamma$ is the intersection of the x-y plane with S^2, i.e. the equator. Earlier we saw that $f \circ \gamma$ is a geodesic on S^2. So γ must also be a geodesic on S^2.

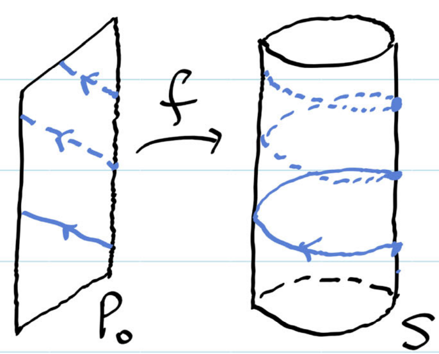


Example. Fix $R>0$. Let $S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = R^2\}$ be a cylinder in $\mathbb{R}^3$. It is locally isometric to the plane $P = \{(x,y,z) \in \mathbb{R}^3 : x = 0\}$: if $S_0 = \{(x,y,z) \in S : x \neq -R\}$ and $P_0 = \{(x,y,z) \in P : |y| < \pi R\}$, then the map $f: P_0 \to S_0$ defined by $f(0,y,z) = (R\cos\frac{y}{R}, R\sin\frac{y}{R}, z)$ is an isometry from P_0 to S_0. (This map "folds" P_0 up isometrically to give S_0). A curve $\gamma(t)$ on S_0 is a geodesic on S_0, if and only if $f^{-1}\circ\gamma(t)$ is a geodesic in P_0, i.e. if and only if $f^{-1}\circ\gamma(t)$ parametrizes a straight line in P_0 at constant speed. This way we determine all geodesics on S_0, and hence on S by rotation invariance. Here are all geodesics on S (all at constant speeds).

① horizontal circles ② vertical straight lines ③ helices





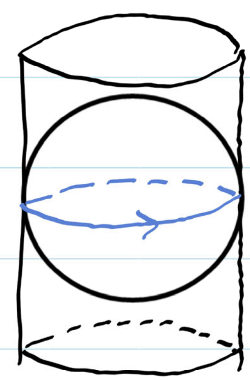


Here is another useful trick for finding geodesics on regular surfaces.

Proposition. Let S_1, S_2 be regular surfaces that are tangent to each other along a common curve γ. Then γ is a geodesic on S_1, if and only if γ is a geodesic on S_2.

Proof. The tangent plane to S_1 at $\gamma(t)$ is the same as the tangent plane to S_2 at $\gamma(t)$. Hence the orthogonal projections of $\frac{d\gamma}{dt}$ onto $T_{\gamma(t)}(S_1)$ is the same as that onto $T_{\gamma(t)}(S_2)$, i.e. the covariant derivative of $\frac{d\gamma}{dt}$ on S_1 is the same as the covariant derivative on S_2. Hence γ is a geodesic on S_1, if and only if γ is a geodesic on S_2.

Example. A simple way to show that the equator of a sphere is a geodesic on the sphere is to observe that a cylinder is tangent to the sphere along its equator, and that the equator to the sphere is a geodesic on that cylinder.



In general, if S is a regular surface in $\mathbb{R}^3$, $p \in S$, and $v \in T_p(S)$, then theory of ordinary differential equations show that the geodesic equation $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$ with initial conditions $\gamma(0) = p$ and $\gamma'(0) = v$ is locally solvable in t (for some possibly very short, but positive amount of time). So for any $p \in S$, and any $v \in T_p(S)$, there exists $\varepsilon > 0$ and some geodesic $\gamma : (-\varepsilon, \varepsilon) \to S$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. In practice the equation $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$ is a second order non-linear system of ordinary differential equations (with 2 equations and 2 unknowns), and is usually not easy to solve. This is why we had all the previous tricks finding geodesics.

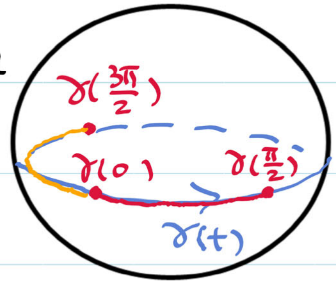
Next we mention 3 properties of geodesics.

Proposition. If $\gamma(t)$ is a geodesic on a regular surface S, then its speed $|\gamma'(t)|$ is constant independent of t.

Proof. $\gamma'(t) = \frac{d\gamma}{dt}$ is a parallel vector field along $\gamma(t)$.
Hence by a property of parallel vector fields, $|\gamma'(t)|$ is constant.

Proposition. If $\gamma(t)$ is a geodesic on a regular surface S, then for t_1 sufficiently close to t_2, the restriction of γ to $[t_1, t_2]$ is the shortest curve on S connecting $\gamma(t_1)$ and $\gamma(t_2)$ (in other words, any other curve on S that connects $\gamma(t_1)$ to $\gamma(t_2)$ is strictly longer).
We say that "geodesics are locally length minimizing".

Example Let $\gamma(t) = (\cos t, \sin t, 0)$ be a geodesic on S^2.
If $t_1 = 0$, $t_2 = \frac{\pi}{2}$, then $\gamma|_{[0, \frac{\pi}{2}]}$ is the shortest curve on S^2 connecting $\gamma(0)$ to $\gamma(\frac{\pi}{2})$, but this fails if say $t_1 = 0$, $t_2 = \frac{3\pi}{2}$: $\gamma|_{[0, \frac{3\pi}{2}]}$ is not the shortest curve on S^2 from $\gamma(0)$ to $\gamma(\frac{3\pi}{2})$. The orange curve is.



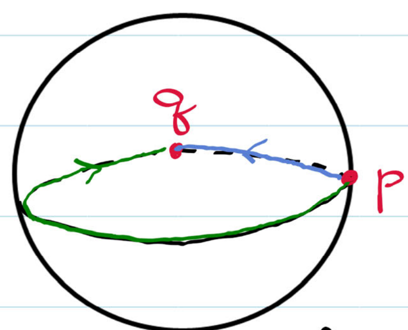
There is also a partial converse to the previous proposition.

Proposition Let p, q be points on a regular surface S, and $\gamma : [a,b] \to S$ be a C^∞ curve on S with $\gamma(a) = p$ and $\gamma(b) = q$. If γ is the shortest such curve on S, then γ is a geodesic.

(In this case, we define the distance between p and q to be the length of γ. Proof of proposition omitted.)

Example. Let S = unit sphere $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ and $p = (1,0,0)$, $q = (0,1,0)$. Find the distance between p and q on S.

Solution. The geodesics on S are precisely the great circles. Hence there are 2 (simple) geodesics joining p to q, namely



$\gamma_1(t) = (\cos t, \sin t, 0)$, $t \in [0, \frac{\pi}{2}]$, and

$\gamma_2(t) = (\cos t, -\sin t, 0)$, $t \in [0, \frac{3\pi}{2}]$. Since $\text{length}(\gamma_1) = \frac{\pi}{2} < \frac{3\pi}{2} = \text{length}(\gamma_2)$, the distance between p and q on S is $\frac{\pi}{2}$.

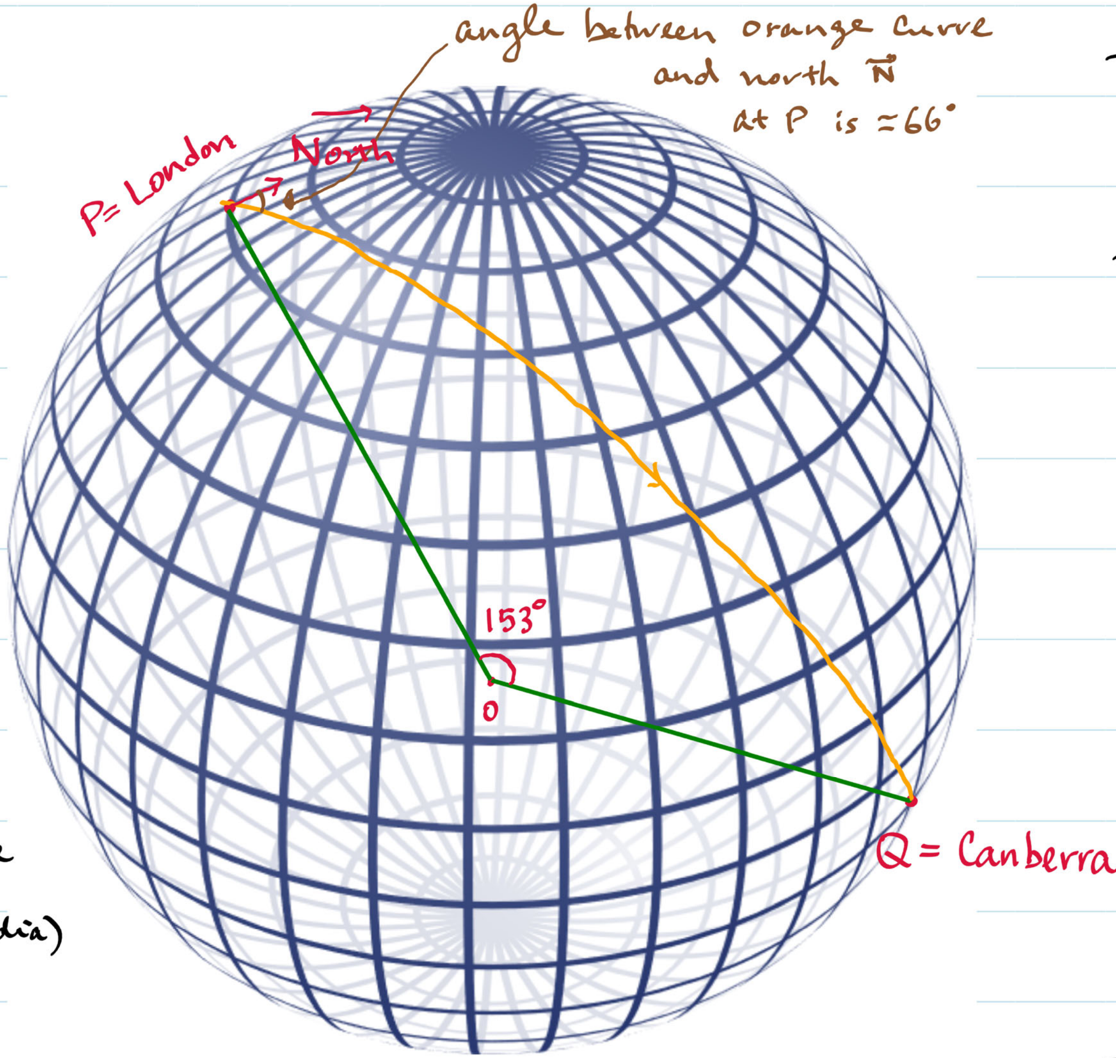
Example. Assume that the Earth is a round sphere of radius 6371 km and the longitudes and latitudes of Canberra and London are

	longitude	latitude
Canberra	149° E	35° S
London	0° E	51° N

Find the length of the shortest path on Earth between the two cities.

Solution. Parametrize Earth by $\underset{\sim}{x}(\theta,\phi) = (6371\cos\theta\cos\phi,\ 6371\sin\theta\cos\phi,\ 6371\sin\phi)$ so that London corresponds to $P = \underset{\sim}{x}(0°, 51°) \simeq (4009, 0, 4051)$ and Canberra corresponds to $Q = \underset{\sim}{x}(149°, -35°) \simeq (-4473, 2688, -3654)$. Let $O = (0,0,0)$ be the centre of Earth, and Σ be the plane in $\mathbb{R}^3$ passing through O, P and Q. Then the shortest path on Earth from P to Q lies on the intersection of Σ with the Earth; indeed it is a circular arc of radius 6371 km substending angle $\angle POQ \in (0, 180°)$. But $\angle POQ = \cos^{-1}\left(\frac{\vec{OP}\cdot\vec{OQ}}{|\vec{OP}||\vec{OQ}|}\right) = \cos^{-1}\left(\frac{(4009,0,4051)\cdot(-4473,2688,-3654)}{6371^2}\right)$ $\simeq 2.66268$ radians $(\simeq 153°)$, so the required length is $6371 \times 2.66268 \text{ km} \simeq 16964 \text{km}$.

because such shortest path is a geodesic!



(Picture from Wikipedia)

The orange path represents the shortest path from London to Canberra. Remarkably, even though Canberra is both south and east of London, the shortest path from London to Canberra starts by going northeast (066°), not southeast!

Indeed, a normal direction to the plane Σ is given by $\dfrac{\vec{OP}\times\vec{OQ}}{|\vec{OP}\times\vec{OQ}|} \simeq (-0.71, -0.40, 0.58)$, and the unit tangent vector to the orange curve is $(-0.71, -0.40, 0.58)\times\dfrac{\vec{OP}}{|\vec{OP}|} = (-0.31, 0.92, 0.25)$, which makes angle $\simeq 66°$ with north direction at P.

$\vec{OP} = (4009, 0, 4051)$, $\vec{OQ} = (-4473, 2688, -3654)$.

$\vec{North} = \dfrac{X_\phi(0°, 51°)}{|X_\phi(0°, 51°)|} = (-0.78, 0, 0.63)$ is pointing north at P = London.