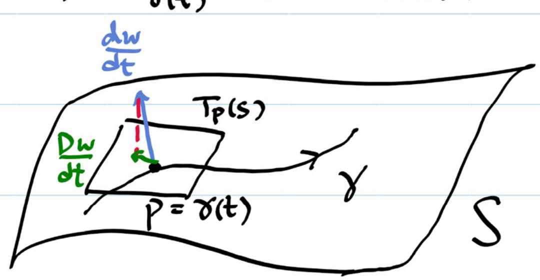
Geodesic curvature

Let S be an oriented regular surface in $\mathbb{R}^3$, with a continuous choice of unit normal N on S. Let $\gamma(s)$ be an arc-length parametrized C^∞ curve on S, i.e. $\gamma(s) \in S$ and $|\gamma'(s)| = 1$ for every s. Previously we defined its curvature $k(s) = |\gamma''(s)|$, and defined its normal curvature $k_n(s) = \gamma''(s) \cdot N$ (where by N we mean $N(\gamma(s))$). Today we introduce its geodesic curvature, denoted by $k_g(s)$. In fact, we will first consider a slightly more general setting as follows.

Let $\gamma(t)$ be any C^∞ curve on S (not necessarily parametrized by arc length now) and let $w(t)$ be a C^∞ unit vector field along $\gamma(t)$, i.e. $w(t) \in T_{\gamma(t)}(S)$ $\forall t$ and $|w(t)|=1$.

Earlier we defined its covariant derivative $\frac{Dw}{dt}$ along γ:

$\frac{Dw}{dt}$ = the orthogonal projection of $\frac{dw}{dt}$ to $T_{\gamma(t)}(S)$.



Today, first note that N and w are unit vectors that are orthogonal to each other. If $\bar{w} := N \wedge w$, then $\{N, w, \bar{w}\}$ form an orthonormal basis with positive orientation in $\mathbb{R}^3$. We define the algebraic value $\left[\frac{Dw}{dt}\right]$ of the covariant derivative $\frac{Dw}{dt}$, by $\left[\frac{Dw}{dt}\right] = \frac{Dw}{dt} \cdot \bar{w}$ $\left(= \frac{Dw}{dt} \cdot (N \wedge w)\right)$. Since $\frac{dw}{dt} - \frac{Dw}{dt}$ is a multiple of N, and $\bar{w}$ is normal to N, we also have $\left[\frac{Dw}{dt}\right] = \frac{dw}{dt} \cdot \bar{w}$ $\left(= \frac{dw}{dt} \cdot (N \wedge w)\right)$. Since $w \cdot w = 1$ $\forall t$, differentiating this and using $\frac{d}{dt}(w \cdot w) = 2 \frac{Dw}{dt} \cdot w$ gives $\frac{Dw}{dt} \cdot w = 0$, and by construction $\frac{Dw}{dt} \cdot N = 0$. Altogether, we see that in the orthonormal frame $\{N, w, \bar{w}\}$, we have $\frac{Dw}{dt} = \left[\frac{Dw}{dt}\right] \bar{w} = \left[\frac{Dw}{dt}\right] N \wedge w$; in particular, the absolute value of $\left[\frac{Dw}{dt}\right]$ is the length of the vector $\frac{Dw}{dt}$.

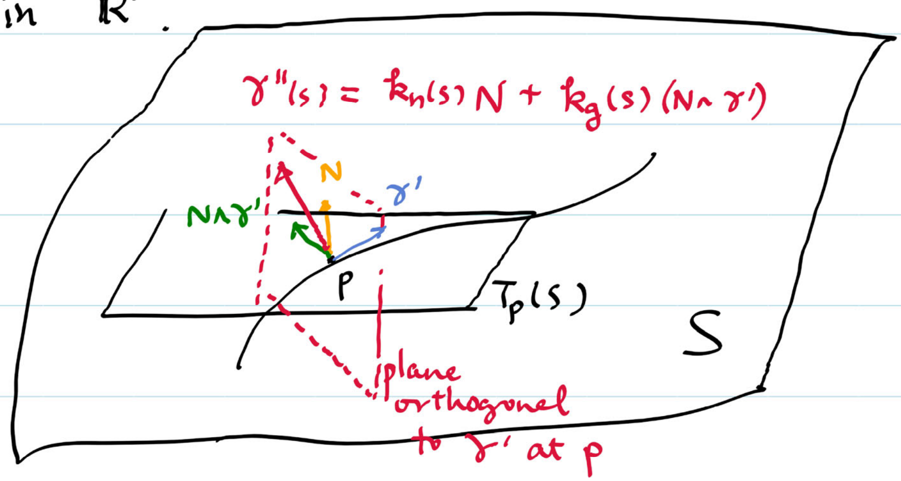
We will next apply this to $w := \gamma'(s)$, where $\gamma(s)$ is a C^∞ arc-length parametrized curve on an oriented regular surface S. In that case, we define the geodesic curvature $k_g(s)$ of $\gamma(s)$ to be $k_g(s) := \left[\frac{D}{ds}\left(\frac{d\gamma}{ds}\right)\right]$ (so $k_g(s) = \frac{d^2\gamma}{ds^2} \cdot \left(N \wedge \frac{d\gamma}{ds}\right) = \gamma''(s) \cdot (N \wedge \gamma'(s))$ from the discussion above).

From our earlier discussion, we also have $\frac{D}{ds}\left(\frac{d\gamma}{ds}\right) = k_g(s)\left(N \wedge \frac{d\gamma}{ds}\right)$;
in particular, $\gamma(s)$ is a geodesic, if and only if the geodesic curvature $k_g(s)$ vanishes identically.

Example. Let $\underline{x}(u,v) = (\cos u \cos v, \cos u \sin v, \sin u)$ be a local parametrization of the unit sphere S^2 and $\gamma(s) = \underline{x}\left(\phi, \frac{s}{\cos\phi}\right)$ be a latitude parametrized at unit speed where $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is fixed.
Find the geodesic curvature of $\gamma(s)$.

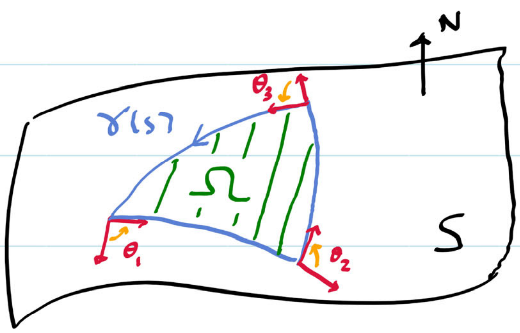
Solution. Note $\frac{d\gamma}{ds} = \frac{1}{\cos\phi}\underline{x}_v = (-\sin v, \cos v, 0)$, and $N = (\cos u \cos v, \cos u \sin v, \sin u)$, so $N \wedge \frac{d\gamma}{ds} = (-\sin u \cos v, -\sin u \sin v, \cos u) = \underline{x}_u$. From here, we may either recall we calculated $\Gamma_{22}^1 = \sin u \cos u$ and $\Gamma_{22}^2 = 0$, so that $\frac{D}{ds}\left(\frac{d\gamma}{ds}\right)$
$= \frac{1}{\cos\phi}\nabla_{\underline{x}_v}\left(\frac{1}{\cos\phi}\underline{x}_v\right) = \frac{1}{\cos^2\phi}\nabla_{\underline{x}_v}\underline{x}_v = \frac{1}{\cos^2\phi}\sin\phi\cos\phi\,\underline{x}_u = \tan\phi\,\underline{x}_u$, which implies $k_g(s) = \frac{D}{ds}\left(\frac{d\gamma}{ds}\right)\cdot\left(N \wedge \frac{d\gamma}{ds}\right) = \tan\phi$; or use $\frac{d^2\gamma}{ds^2} = \frac{1}{\cos^2\phi}\underline{x}_{vv}$
$= \frac{1}{\cos\phi}(-\cos v, -\sin v, 0)$ so $k_g(s) = \left[\frac{D}{ds}\left(\frac{d\gamma}{ds}\right)\right] = \frac{d^2\gamma}{ds^2}\cdot\left(N \wedge \frac{d\gamma}{ds}\right) = \frac{\sin\phi}{\cos\phi} = \tan\phi$.
Either way we get the same answer $k_g(s) = \tan\phi$. (Again, $\gamma(s)$ is geodesic iff $\tan\phi = 0$, i.e. $\phi = 0$).

Let $\gamma(s)$ be a C^∞ arc-length parametrized curve on a regular surface S oriented with a continuous unit normal N on S. Earlier we saw that the curvature of $\gamma(s)$ is $k(s) := |\gamma''(s)|$, and the normal curvature $k_n(s) := \gamma''(s) \cdot N$. Just now we defined the geodesic curvature $k_g(s)$ so that $k_g(s) = \gamma''(s) \cdot (N \wedge \gamma'(s))$. Since $\gamma''(s) \cdot \gamma'(s) = 0$ (obtained by differentiating $\gamma'(s) \cdot \gamma'(s) = 1 \quad \forall s$), and since $\{N, \gamma', N \wedge \gamma'\}$ is an orthonormal frame in $\mathbb{R}^3$, we have now $\gamma''(s) = k_n(s) N + k_g(s)(N \wedge \gamma')$, and Pythagoras's theorem shows that $k(s) = \sqrt{k_n(s)^2 + k_g(s)^2}$. This completes our study of various curvatures of C^∞ arc-length parametrized curves $\gamma(s)$ on a regular surface S in $\mathbb{R}^3$.



The local Gauss-Bonnet theorem

Let S be an oriented regular surface in $\mathbb{R}^3$, and let $\Omega \subseteq S$ be a region homeomorphic to a closed disc in $\mathbb{R}^2$. Assume the boundary of Ω is parametrized as a simple, closed, piecewise C^∞, positively oriented curve $\gamma(s)$ parametrized by arc length.



[simple: no self intersection;

closed: end-point = starting point;

piecewise C^∞: C^∞ curves joined tip to end;

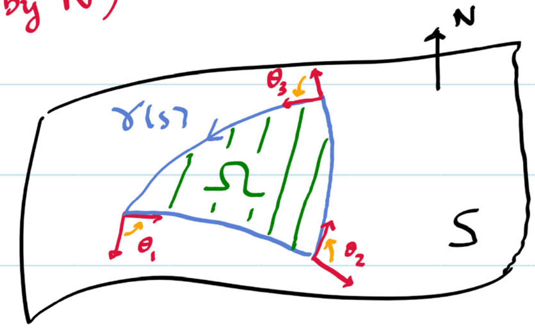
positively oriented: If you walk along $\gamma(s)$ with your head up in the direction of N, then Ω is on your left side.]

Let $\theta_1, \dots, \theta_N \in [-\pi, \pi]$ be the angle turned at each point where γ is not C^∞ (keep your head up at such a point in direction N, and measure θ_i as positive if you turn counterclockwise, negative if you turn clockwise).

The local Gauss-Bonnet theorem asserts that if $K =$ Gaussian curvature on S, then

$$\underbrace{\iint_\Omega K\, dA}_{\substack{\text{surface integral}\\ \text{of Gaussian}\\ \text{curvature on } \Omega}} + \underbrace{\int_\gamma k_g\, ds}_{\substack{\text{path integral}\\ \text{of geodesic}\\ \text{curvature on } \gamma}} + \underbrace{\sum_i \theta_i}_{\substack{\text{sum of angles}\\ \text{turned around } \gamma}} = 2\pi$$

(in the orientation determined by N)



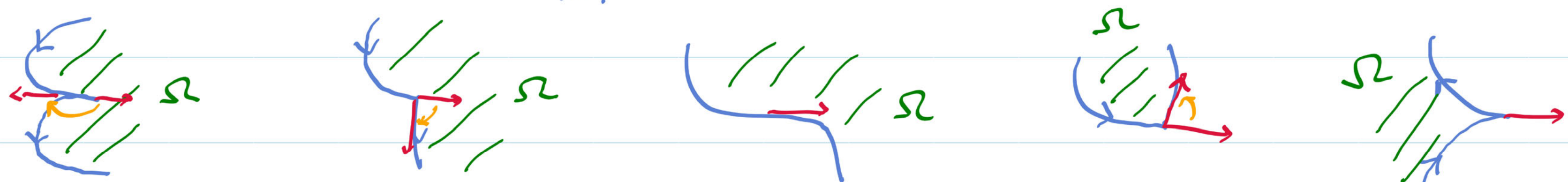
Without loss of generality we may assume that $\Omega = \underline{x}(R)$ where $\underline{x} : U \to \underline{x}(U) \subseteq S$ is some local parametrization, and $R \subseteq U$. Then

$$\iint_\Omega K\, dA = \iint_R K(\underline{x}(u,v))\, |\underline{x}_u \wedge \underline{x}_v|\, du\, dv = \iint_R K(\underline{x}(u,v)) \sqrt{EG - F^2}\, du\, dv$$

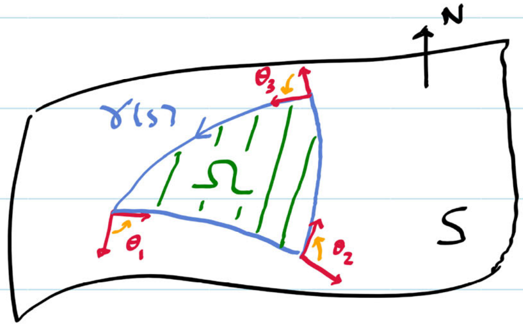
where E, F, G are the coefficients of the first fundamental form on $\underline{x}(U)$.

Furthermore, $\displaystyle\int_\gamma k_g\, ds = \sum_{i=1}^N \int_{a_{i-1}}^{a_i} k_g(\gamma(s))\, ds$. Some examples of $\theta \in [-\pi, \pi]$:

$\odot$ N pointing out of screen



$\theta = -\pi$ $\quad -\pi < \theta < 0$ $\quad \theta = 0$ $\quad 0 < \theta < \pi$ $\quad \theta = \pi$



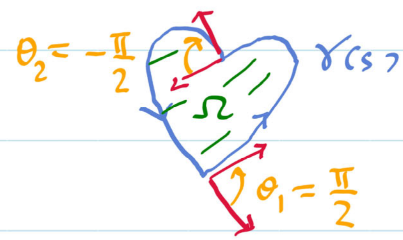
$$\underbrace{\iint_\Omega K\, dA}_{\substack{\text{surface integral} \\ \text{of Gaussian} \\ \text{Curvature on } \Omega}} + \underbrace{\int_\gamma k_g\, ds}_{\substack{\text{path integral} \\ \text{of geodesic} \\ \text{Curvature on } \gamma}} + \underbrace{\sum_i \theta_i}_{\substack{\text{sum of angles} \\ \text{turned around } \gamma \\ \text{(in the orientation} \\ \text{determined by } N)}} = 2\pi$$

e.g. If S = plane (so the Gaussian Curvature $K \equiv 0$ and k_g = curvature $k(s)$ of the plane curve $\gamma(s)$), then this formula says that

$$\int_\gamma k(s)\, ds + \sum_i \theta_i = 2\pi$$

(whenever γ is a simple, closed, piecewise C^∞ curve parametrized by arc length and has counter-clockwise orientation)

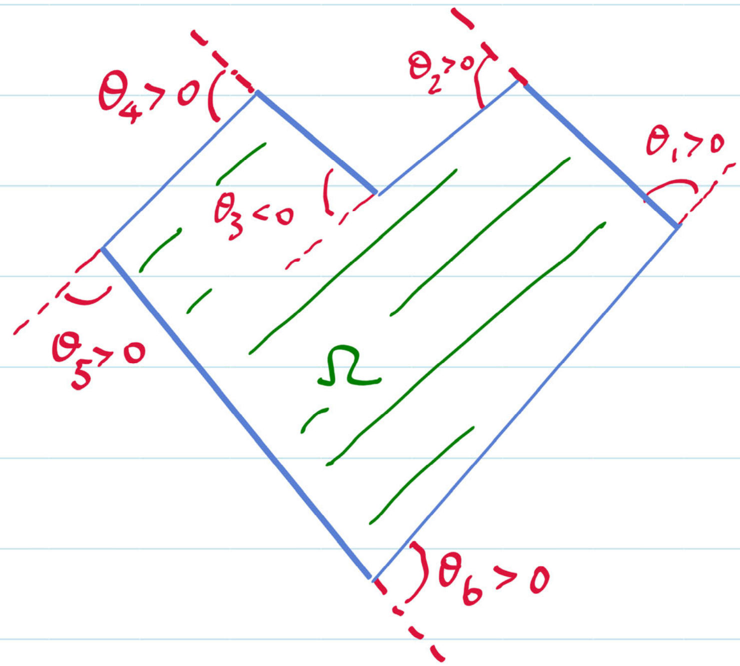
Then $\int_\gamma k(s)\, ds + \sum_i \theta_i = 2\pi$



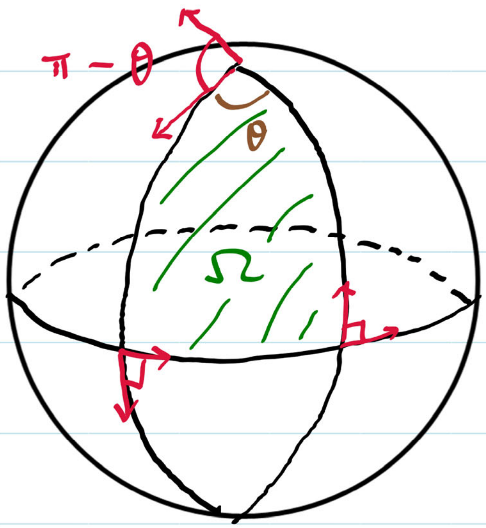
In this picture, $\int_\gamma k(s)\, ds = 2\pi$.

e.g. If not only $S =$ plane, but $\Omega =$ polygon in the plane, then the local Gauss-Bonnet theorem says $\sum_i \theta_i = 2\pi$

(sum of exterior angles $= 2\pi$)



e.g. Orient a sphere of radius R with the outer unit normal. Let Ω be the "triangle" formed by 3 geodesics, namely the equator and two longitudes that are θ degrees apart.



Gaussian curvature $K = \frac{1}{R^2}$

$$\Rightarrow \iint_\Omega K \, dA = \frac{\text{Area}(\Omega)}{R^2}$$

$$\sum_i \theta_i = \frac{\pi}{2} + \frac{\pi}{2} + (\pi - \theta) = 2\pi - \theta$$

Then the local Gauss-Bonnet says $\iint_\Omega K \, dA + \sum_i \theta_i = 2\pi$,

i.e. $\frac{\text{Area}(\Omega)}{R^2} + (2\pi - \theta) = 2\pi$, i.e. $\text{Area}(\Omega) = R^2\theta$

Sketch of proof of local Gauss-Bonnet: Use Green's theorem.

First, without loss of generality, assume $\Omega = \underline{x}(R)$ for some local parametrization $\underline{x}$ and some closed and bounded region R in the domain of $\underline{x}$. We may also assume $F \equiv 0$ on $\underline{x}(R)$. Then the Gaussian curvature K at $\underline{x}(u,v)$ is given by

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v\right].$$

(Assignment 5)

Green's theorem: $\iint_R (P_u + Q_v)\,du\,dv = \int_{\partial R} P\,dv - Q\,du$

∂R: positively oriented

Hence
$$\iint_\Omega K\,dA = \iint_R -\frac{1}{2\sqrt{EG}}\left[\left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v\right]\sqrt{EG-0^2}\,du\,dv$$
$$= -\iint_R \frac{1}{2}\left[\left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v\right]du\,dv$$
$$= -\int_{\partial R}\frac{1}{2}\left(\frac{G_u}{\sqrt{EG}}dv - \frac{E_v}{\sqrt{EG}}du\right) = -\int_\gamma \frac{1}{2}\left(\frac{G_u}{\sqrt{EG}}\frac{dv}{ds} - \frac{E_u}{\sqrt{EG}}\frac{du}{ds}\right)ds$$

where $\gamma(s)$ is a parametrization of ∂R by arc length.

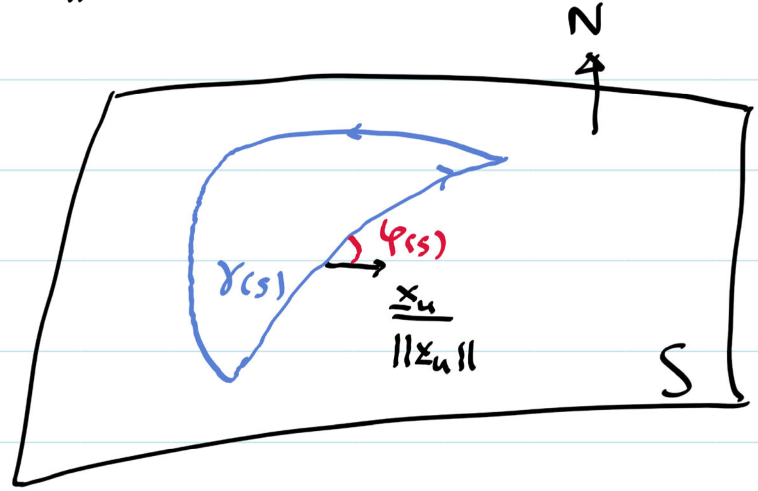
But $\frac{1}{2}\left(\frac{G_u}{\sqrt{EG}}\frac{dv}{ds} - \frac{E_v}{\sqrt{EG}}\frac{du}{ds}\right) = \left[\frac{D}{ds}\left(\frac{\underline{x}_u}{\|\underline{x}_u\|}\right)\right]$ at every point where γ is C^∞.

So $\iint_\Omega K\, dA = -\int_\gamma \left[\frac{D}{ds}\left(\frac{x_u}{\|x_u\|}\right)\right] ds.$

It can be shown that $\left[\frac{D}{ds}\left(\frac{x_u}{\|x_u\|}\right)\right] = \overbrace{\left[\frac{D}{ds}\left(\frac{d\gamma}{ds}\right)\right]}^{k_g(s)} - \frac{d\varphi}{ds}$

where $\varphi(s)$ is the angle from $\frac{x_u}{\|x_u\|}$ to $\frac{d\gamma}{ds}$ in the orientation given by N. Hence

$$\iint_\Omega K\, dA + \int_\gamma k_g(s)\, ds = \int_\gamma \frac{d\varphi}{ds}\, ds,$$



and it remains to show that

$$\int_\gamma \frac{d\varphi}{ds}\, ds = 2\pi - \sum_i \theta_i,$$

whose proof involves topology (the assumption that Ω is homeomorphic to a closed disc) which we omit.

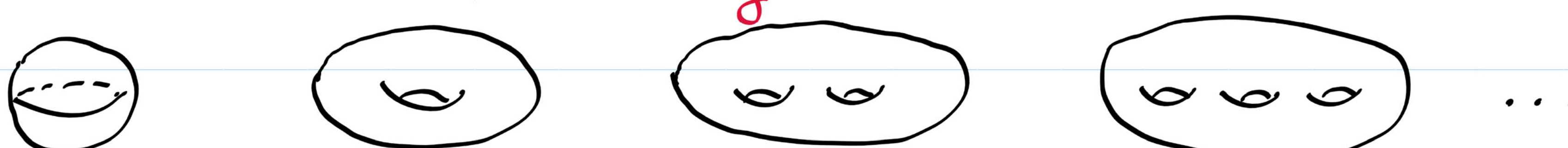
"Theorem of turning tangents"

Classification of compact orientable surfaces in $\mathbb{R}^3$.

A subset of $\mathbb{R}^3$ is said to be **compact** if it is closed and bounded. It turns out that any compact, orientable surfaces in $\mathbb{R}^3$ is homeomorphic to exactly one of the following surfaces:

sphere "no hole" | torus "1 hole" | "2 holes" | "3 holes"

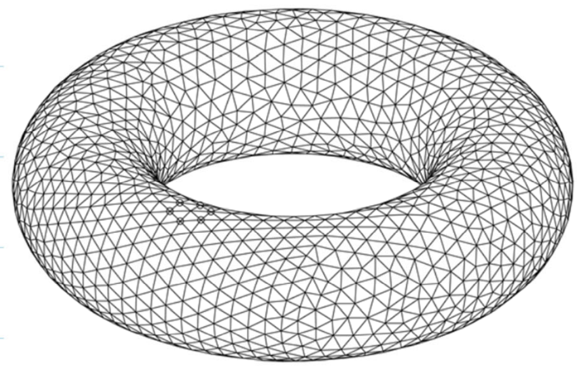
The number of "holes" in such a surface is called the **genus** g of the surface. The **Euler characteristic** of such a surface S is defined to be $\chi(S) = 2-2g$.



$g=0,\ \chi=2$ | $g=1,\ \chi=0$ | $g=2,\ \chi=-2$ | $g=3,\ \chi=-4$

If we triangulate such a surface S (by "tiling" it with "triangles", where each "triangle" is a region on the surface whose boundary is the union of 3 C^∞ curves joined tip to end), then it can be shown that

$$V - E + F = \chi(S)$$



where V = number of vertices

E = number of edges

F = number of faces

in this triangulation.

(Source: wikipedia)

e.g. Imagine cutting an orange into 8 pieces, by cutting exactly once along the 3 coordinate planes. Then the surface of the orange is cut into 8 "triangles". In this case, $V = 6$ (2 at the poles and 4 on the equator), $E = 12$ (4 on each hemisphere and 4 on equator), $F = 8$ (8 triangles), and we can check that $V - E + F = 2 = \chi(S^2)$. (Remarkably, this holds for any other triangulations!)

The global Gauss-Bonnet theorem

The global Gauss-Bonnet theorem states that for any compact, orientable regular surface S, we have $\iint_S K\,dA = 2\pi\chi(S)$ where the left hand side is the surface integral of the Gaussian curvature K on S.

e.g. If S = sphere of radius R, then $K = \frac{1}{R^2}$ at every point,
so $\iint_S K\,dA = \frac{1}{R^2}\cdot \text{Area}(S) = \frac{1}{R^2}\cdot 4\pi R^2 = 4\pi = 2\pi\,\chi(S^2)$,
as asserted by the global Gauss-Bonnet theorem.

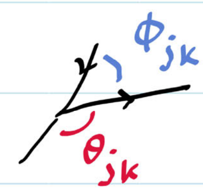
But the global Gauss-Bonnet theorem says much more.
e.g. It also asserts that if S is <u>any</u> ellipsoid in $\mathbb{R}^3$,
then $\iint_S K\,dA = 4\pi$ (because then $\chi(S) = 2$).

To appreciate the global Gauss-Bonnet theorem a bit more, note that the assertion that $\iint_S K\,dA = 2\pi\,\chi(S)$ holds for <u>all</u> Compact orientable regular surfaces S is a very remarkable one, because there are so many compact orientable regular surfaces S, and their Gaussian curvatures K can behave in many different ways. In particular, K depends on the geometry of S, but the global Gauss-Bonnet theorem asserts that its integral over S is a topological invariant: remember $\chi(S)$ depends only on the topology of S, but not on the geometry of S.

An application: If S is a compact, orientable regular surface in $\mathbb{R}^3$ and $\iint_S K\,dA > 0$, then $\iint_S K\,dA$ must indeed be equal to 4π (because $\chi(S) = 2-2g$ is positive only when $g=0$, in which case $\chi(S)=2$ and $\iint_S K\,dA = 2\pi\,\chi(S) = 4\pi$), in which case S must be homeomorphic to S^2 (by our earlier classification of compact orientable surfaces).

Sketch of proof of the global Gauss-Bonnet theorem: triangulate a compact orientable regular surface S and apply local Gauss-Bonnet on each "triangle". Note that if γ is a "side" of such "triangle", and $\tilde{\gamma}$ is γ transversed in the opposite orientation, then writing k_g and $\tilde{k}_g$ for the geodesic curvatures of γ and $\tilde{\gamma}$ respectively, we have $k_g = -\tilde{k}_g$, so $\int_\gamma k_g \, ds = -\int_{\tilde{\gamma}} \tilde{k}_g \, ds$. As a result, if $\{T_j\}$ is a triangulation of S, then $\sum_j \int_{\partial T_j} k_g \, ds = 0$. Now we appeal to the local Gauss-Bonnet on each T_j: we get

$$(*) \qquad \iint_{T_j} K \, dA + \int_{\partial T_j} k_g \, ds + (\theta_{j1} + \theta_{j2} + \theta_{j3}) = 2\pi$$



Write $\theta_{jk} = \pi - \phi_{jk}$ for $k = 1, 2, 3$, and note sum of the ϕ's at any vertex of this triangulation is 2π. So now sum $(*)$ over j: we get $\iint_S K \, dA + 3\pi F - 2\pi V = 2\pi F$. But each edge is shared between exactly two faces, so $3F = 2E$, and we get

$\iint_S K \, dA = 2\pi (V - E + F) = 2\pi \chi(S)$, as desired.