MATH2320/MATH3116/MATH6110 Homework 1

Due Date: 7 May 2020.

[Marking scheme: 112 points + 13 points for presentation. Total: 125 points.]
Please submit your work via Wattle in one single pdf file.

1. In class we defined real numbers to be equivalent classes of Cauchy sequences of rationals. Using what we discussed in class, determine whether each of the following statements is true or false. Explain your answer. (8 points each)

   (a) Let (r_n) be a Cauchy sequence of rationals and let x be the equivalent class containing the sequence (r_n). Then $x > 0$ if and only if there exists $N \in \mathbb{N}$ such that $r_n > 0$ for every $n \geq N$.

   (b) Let x be an equivalent class of Cauchy sequences of rationals. Then $x > 0$ if and only if for any sequence (r_n) in the equivalent class x, there exists $N \in \mathbb{N}$ such that $r_n > 0$ for every $n \geq N$.

   (c) Let (r_n), (s_n) be Cauchy sequences of rationals and let x, y be the equivalent classes containing (r_n) and (s_n) respectively. Then $x \leq y$, if and only if for every $\varepsilon \in \mathbb{Q}_+$, there exists $N \in \mathbb{N}$ such that $r_n \leq s_n + \varepsilon$ whenever $n \geq N$.

   (d) Let (r_n) be a Cauchy sequence of rationals. Then there exists a Cauchy sequence of rationals (s_n), equivalent to (r_n), such that $|s_n - s_m| \leq 2^{-\min\{n,m\}}$ for all $m, n \in \mathbb{N}$.

2. Let $p \in \mathbb{N}$ be a prime number. Define the p-adic absolute value on $\mathbb{Q}$ as follows. Let $|0|_p := 0$ and for $r \in \mathbb{Q} \setminus \{0\}$, let

$$|r|_p := p^{-m} \quad \text{if } r = \frac{p^m a}{b} \text{ where } m \in \mathbb{Z},\ a, b \in \mathbb{Z} \setminus \{0\}, \text{ and } p \nmid a,\ p \nmid b.$$

   (Here $p \nmid a$ means a is not an integer multiple of p.) It is known that $|\cdot|_p$ satisfies

$$\begin{aligned} |r|_p &\geq 0 \quad \text{for all } r \in \mathbb{Q}, \text{ with equality if and only if } r = 0, \\ |rs|_p &= |r|_p|s|_p \quad \text{for all } r, s \in \mathbb{Q}, \text{ and} \\ |r + s|_p &\leq \max\{|r|_p, |s|_p\} \quad \text{for all } r, s \in \mathbb{Q}; \end{aligned}$$

   in particular, the latter implies the triangle inequality, namely

$$|r + s|_p \leq |r|_p + |s|_p$$

   for every $r, s \in \mathbb{Q}$. A sequence of rational numbers (r_n) is said to be p-Cauchy, if for every $\varepsilon \in \mathbb{Q}_+$, there exists $N \in \mathbb{N}$ such that $|r_n - r_m|_p < \varepsilon$ whenever $n \geq N$. Two p-Cauchy sequences of rationals (r_n) and (s_n) are said to be equivalent, if for every $\varepsilon \in \mathbb{Q}_+$, there exists $N \in \mathbb{N}$ such that $|r_n - s_n|_p < \varepsilon$ whenever $n \geq N$.

   (a) (10 points) Show that if (r_n) is a p-Cauchy sequence of rationals and (r_n) is not equivalent to the zero sequence (0), then there exists $N \in \mathbb{N}$ such that $|r_n|_p = |r_N|_p$ whenever $n \geq N$.

   (b) Let $\mathbb{Q}_p$ be the set of all equivalent classes of p-Cauchy sequences of rationals.

      (i) (10 points) Define addition and multiplication on $\mathbb{Q}_p$ to make it a field. You should check that the operations you defined are well-defined, state what the additive identity and the multiplicative identities are, and explain why additive and multiplicative inverses exist in $\mathbb{Q}_p$ when they exist. But you are not required to check the remaining field axioms.

      (ii) (10 points) If $x \in \mathbb{Q}_p$, define

$$|x|_p := \lim_{n \to \infty} |r_n|_p$$

      where (r_n) is any p-Cauchy sequence of rationals in the equivalent class x. Show that this is well-defined on $\mathbb{Q}_p$, and that for $x \in \mathbb{Q}_p$, we have $|x|_p = 0$ if and only if x is the class equivalent to the zero sequence (0).

(c) (16 points) A sequence (x_j) in $\mathbb{Q}_p$ is said to be Cauchy, if for every $\varepsilon \in \mathbb{Q}_+$, there exists $J \in \mathbb{N}$ such that $|x_j - x_k|_p < \varepsilon$ whenever $j, k \geq J$. A sequence (x_j) in $\mathbb{Q}_p$ is said to converge to $x \in \mathbb{Q}_p$, if for every $\varepsilon \in \mathbb{Q}_+$, there exists $J \in \mathbb{N}$ such that $|x_j - x|_p < \varepsilon$ whenever $j \geq J$. Show that every Cauchy sequence $(x_j) \in \mathbb{Q}_p$ converges to some $x \in \mathbb{Q}_p$. (Hint: Question 1(d) may be useful here.)

(d) (10 points) Let (r_n) be the sequence of rationals defined by

$$r_n := 1 + p + p^2 + \cdots + p^n.$$

Show that there exists $r \in \mathbb{Q}$ so that (r_n) converges to r in $\mathbb{Q}_p$. Here we consider $\mathbb{Q}$ to be a subset of $\mathbb{Q}_p$, via the embedding $r \in \mathbb{Q} \mapsto [(r, r, r, \dots)] \in \mathbb{Q}_p$.

3. (10 points) Define a norm on $\mathbb{Q}$ by

$$|r| := \begin{cases} 1 & \text{if } r \in \mathbb{Q} \setminus \{0\} \\ 0 & \text{if } r = 0. \end{cases}$$

What is the completion of $\mathbb{Q}$ under the metric d associated with this norm? (In other words, what is the smallest complete metric space that contains $(\mathbb{Q}, d)$ as a metric subspace? A complete metric space is one where every Cauchy sequence converges.) Justify your assertion.

4. (14 points) Solve either part (a) or part (b) of this problem. If you are taking MATH6110, you should solve part (b).

(a) Let $C^0[0,1]$ be the space of complex-valued continuous functions on $[0,1]$, and let

$$\|f\|_{L^2} = \left(\int_0^1 |f(x)|^2 dx \right)^{1/2}$$

be the L^2 norm on $C^0[0,1]$. Also let $\ell^2(\mathbb{Z})$ be the space of complex sequences $(x_n)_{n \in \mathbb{Z}}$ with

$$|(x_n)|_{\ell^2} := \left(\sum_{n=-\infty}^{\infty} |x_n|^2 \right)^{1/2} < \infty.$$

Show that $C^0[0,1]$ is not complete under the L^2 norm, whereas $\ell^2(\mathbb{Z})$ is complete under the ℓ^2 norm.

(b) Let $C^1[0,1]$ be the space of complex-valued C^1 functions on $[0,1]$, i.e. the space of all functions $f\colon [0,1] \to \mathbb{C}$ so that $f = F$ on $[0,1]$ for some function F has a continuous derivative on some open interval containing $[0,1]$. Let

$$\|f\|_{W^{1,2}} = \left(\int_0^1 |f(x)|^2 + |f'(x)|^2 dx \right)^{1/2}$$

be the $W^{1,2}$ norm on $C^1[0,1]$. Also let $\ell^{1,2}(\mathbb{Z})$ be the space of complex sequences $(x_n)_{n \in \mathbb{Z}}$ with

$$|(x_n)|_{\ell^{1,2}} := \left(\sum_{n=-\infty}^{\infty} (1 + 4\pi^2 n^2) |x_n|^2 \right)^{1/2} < \infty.$$

Show that $C^1[0,1]$ is not complete under the $W^{1,2}$ norm, whereas $\ell^{1,2}(\mathbb{Z})$ is complete under the $\ell^{1,2}$ norm. (You may use, without proof, the existence of a C^∞ function φ so that $\varphi(x) = 0$ when $x < 1/2$, and $\varphi(x) = 1$ when $x > 1$. You may also assume that $\ell^2(\mathbb{Z})$ is complete.)

Remark. It can be shown that there exists a norm-preserving (hence injective) linear map from $(C^1[0,1], W^{1,2})$ into $\ell^{1,2}(\mathbb{Z})$. So the latter can be taken to be the completion of $C^1[0,1]$ under the $W^{1,2}$ norm. (You are not required to prove this fact.)