MATH2320/MATH3116/MATH6110 Homework 2

Due Date: 5 June 2020.

[Marking scheme: 112 points + 13 points for presentation. Total: 125 points.]
Please submit your work via Wattle in one single pdf file.

1. (14 points) Show that any triangle in the plane with area A has two sides whose lengths have a product that is at least $4A/\sqrt{3}$.
(Hint: How do you express the product of two sides of a triangle in terms of its area and the included angle? There are now a number of ways to finish this problem. For instance, you may take an average over the choice of two sides of the triangle, use that the sum of interior angles of a triangle is π, and then invoke convexity; simpler approaches are also possible.)

2. (14 points) Let $\mathcal{P}$ be a collection of M points and $\mathcal{L}$ be a collection of N lines, all lying in the plane. An incidence between $\mathcal{P}$ and $\mathcal{L}$ is a pair $(p, l) \in \mathcal{P} \times \mathcal{L}$, where l passes through p. Show that the number of incidences between $\mathcal{P}$ and $\mathcal{L}$ is at most $MN^{1/2} + N$.
(Hint: Let $f(p, l) = 1$ if l passes through p, and $f(p, l) = 0$ otherwise. If I denotes the number of incidences between $\mathcal{P}$ and $\mathcal{L}$, then
$$I = \sum_{l \in \mathcal{L}} \sum_{p \in \mathcal{P}} f(p, l).$$
Apply Cauchy-Schwarz to the first sum in l, and then use that any two distinct points on the plane determine a unique line through them.)
Remark. This bound is non-sharp, and may be improved using tools from algebraic topology.

3. (14 points) Suppose $f\colon [0, \infty) \to \mathbb{R}$ is convex, continuously differentiable, and $f(0) = 0$. Show that for any $-1 < \alpha < \infty$ and $A > 0$,
$$\int_0^A x^\alpha \exp\left(-\frac{f(x)}{x}\right) dx \le \exp(\alpha + 1) \int_0^A x^\alpha \exp\left(-f'(x)\right) dx.$$
(Hint: Consider change of variable $x = py$ in the left hand side for p slightly bigger than 1, and compare $f(py)$ to $f(y) + (p-1)yf'(y)$. Apply Hölder, and then let $p \searrow 1$.)

4. (14 points) Show that if $(x_n)_{n=1}^\infty$ is a summable sequence of positive real numbers, then
$$\sum_{n=1}^\infty (x_1 x_2 \cdots x_n)^{1/n} < e \sum_{n=1}^\infty x_n.$$
(Hint: Choose suitable constants $c_1, c_2, \dots$ and apply the AM-GM inequality to show that
$$(x_1 x_2 \cdots x_n)^{1/n} \le \frac{c_1 x_1 + \cdots + c_n x_n}{n} \frac{1}{(c_1 c_2 \dots c_n)^{1/n}}.$$
The $c_1, c_2, \dots$ should be chosen so that one can easily sum over n; it helps to know, for instance, that $\sum_{n=1}^\infty \frac{1}{n(n+1)}$ is readily summable to 1.)

5. (14 points) Show that for any non-negative sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$, one has
$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{x_m y_n}{m+n} \le \pi \left(\sum_{m=1}^\infty x_m^2\right)^{1/2} \left(\sum_{n=1}^\infty y_n^2\right)^{1/2}.$$
This inequality is actually strict if the right hand side of the inequality is finite and strictly positive, but you are not required to prove this; also, you may assume that the right hand side is finite to begin with, for otherwise the inequality is obvious. (Hint: Write
$$\frac{x_m y_n}{m+n} = \frac{x_m}{\sqrt{m+n}} \left(\frac{m^\alpha}{n^\alpha}\right) \cdot \frac{y_n}{\sqrt{m+n}} \left(\frac{n^\alpha}{m^\alpha}\right)$$

for some suitable value of α, and apply Cauchy-Schwarz. It may help to evaluate the integral

$$\int_0^\infty \frac{1}{t+1}\frac{dt}{\sqrt{t}}$$

at some point.)

6. (14 points) Suppose u is a continuously differentiable function on $[0, A]$ for some $A > 0$ and $u(0) = 0$. Show that for $1 < p < \infty$,

$$\left(\int_0^A \left|\frac{u(x)}{x}\right|^p dx\right)^{1/p} \leq \frac{p}{p-1}\left(\int_0^A |u'(x)|^p dx\right)^{1/p}.$$

(Hint: Integrate by parts and apply Hölder.)

7. (14 points) Let k be a positive integer and $a_1, a_2, \dots, a_{2^k}$ be complex numbers. Show that

$$\int_0^1 \max_{1\leq m\leq 2^k}\left|\sum_{n=1}^m a_n e^{2\pi i n x}\right|^2 dx \leq k(k+1)\sum_{n=1}^{2^k}|a_n|^2.$$

(Hint: For $0 \leq j \leq k$, we partition $\{1, 2, 3, \dots, 2^k\}$ into sets of 2^j consecutive integers and call the resulting partition E_j; e.g.

$$E_1 = \{\{1, 2\}, \{3, 4\}, \dots, \{2^k - 1, 2^k\}\}.$$

Let $E := \bigcup_{j=0}^k E_j$. Let m be a positive integer with $1 \leq m \leq 2^k$. How many disjoint sets from E do we need, so that their union is equal to $\{1, 2, 3, \dots, m\}$? How many sets in E contain m? The former question allows one to compare

$$\max_{1\leq m\leq 2^k}\left|\sum_{n=1}^m a_n e^{2\pi i n x}\right|^2 \quad \text{with} \quad \sum_{I\in E}\left|\sum_{n\in I} a_n e^{2\pi i n x}\right|^2.$$

The latter question allows one to estimate the integral of the latter.)

8. (14 points) Let n be a positive integer and let $\gamma\colon [0, 1] \to \mathbb{R}^n$ be the curve given by $\gamma(t) = (t, t^2, \dots, t^n)$ for $t \in [0, 1]$.

   (a) Show that for every positive integer N and every real number $s \geq 1$, the integral

   $$J_{s,N} := \frac{1}{N^{n^2}}\int_{[0,N^n]^n}\left|\sum_{k=1}^N e^{2\pi i\gamma(k/N)\cdot x}\right|^{2s} dx$$

   obeys a lower bound

   $$J_{s,N} \geq \max\{N^s, \frac{1}{2^s(8n)^n}N^{2s-\frac{n(n+1)}{2}}\}$$

   (Hint: For the lower bound N^s, investigate how $J_{s,N}$ varies with s by using Hölder's inequality. $J_{1,N}$ is particularly easy to compute. For the other lower bound, estimate the integrand from below on the set $X_1 \times X_2 \times \cdots \times X_n$, where for $1 \leq i \leq n$, X_i is the set of all $x_i \in [0, N^n]$ satisfying $x_i - mN^i \in [0, \frac{1}{8n}]$ for some integer m.)

   (b) Show that if s, N are positive integers, then the number of solutions to the Diophantine system

   $$\begin{cases} k_1 + k_2 + \cdots + k_s &= k_{s+1} + k_{s+2} + \cdots + k_{2s} \\ k_1^2 + k_2^2 + \cdots + k_s^2 &= k_{s+1}^2 + k_{s+2}^2 + \cdots + k_{2s}^2 \\ &\vdots \\ k_1^n + k_2^n + \cdots + k_s^n &= k_{s+1}^n + k_{s+2}^n + \cdots + k_{2s}^n \end{cases}$$

   with $(k_1, k_2, \dots, k_{2s}) \in \{1, 2, \dots, N\}^{2s}$ is exactly equal to the integral $J_{s,N}$ defined in part (a). Hence part (a) provides a lower bound for the number of solutions to this Diophantine system; it turns out that this lower bound is essentially sharp as $N \to \infty$.