Analysis 1 Add on

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HN 2.58 Office hours Fri 11am-noon

Topics: ① Construction of real number system

② Incomplete function spaces, eg.
the set of Riemann integrable functions
under L^2 inner product on $[0,1]$.
and possibly Fourier series on $[0,1]$.

③ Power series ($\sum_{n=0}^{\infty} a_n x^n$), and
Construction of functions eg e^x, $\sin x$, $\cos x$

④ Some examples of metric spaces
such as function spaces : C^0, Lipschitz
functions, Hölder continuous functions,
Sobolev spaces ...

⑤ Inequalities, such as Cauchy-Schwarz
inequality, Hölder inequality, Minkowski
inequality ...

§1 Construction of the real number system.

$\mathbb{R}$ is a <u>set</u>, equipped with binary operations $+, \cdot$, and a binary relation $<$, that satisfies certain axioms.

<u>Q</u> How do we know that such a set with such binary operations and relation exist?

For context: consider Russell's paradox
Let $T = \{S \text{ is a set} \mid S \notin S\}$. Suppose T is a set.
Then both $T \in T$ and $T \notin T$ leads to contradiction.
So T cannot be a set!

Indeed, axiomatic set theory laid down rules governing the construction of sets, and "the set of all sets" is not a legal construction. (Math 3343!)

Bottom line: can't just write a list of conditions and form a set!

For this add-on: let's assume that $(\mathbb{Q}, +, \cdot, <)$ has been constructed already, and use it to construct $(\mathbb{R}, +, \cdot, <)$.

[Aside: To construct $\mathbb{Q}$:

First $0 := \phi$ the empty set.

For each set X, define successor of X by

$S(X) := X \cup \{X\}$

eg. $S(0) = \{\phi\}$, and we define this to be 1.

Then $\mathbb{N}$ is the smallest set that satisfies:

① $1 \in \mathbb{N}$

② $\forall x \in \mathbb{N}$, then $S(x) \in \mathbb{N}$.

Define $2 := S(1)$, $3 := S(2)$, etc...

i.e. $2 = \{\phi, \{\phi\}\}$, $3 = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$.

and define $+, \cdot$ on $\mathbb{N}$ using S.

Now $\mathbb{Z}$ is then $\mathbb{N} \times \mathbb{N} / \sim$ where

$(m_1, n_1) \sim (m_2, n_2)$ iff $n_1 + m_2 = n_2 + m_1$ (secretly: this is $n_1 - m_1 = n_2 - m_2$)

So that $\mathbb{N} \hookrightarrow \mathbb{Z}$ via $n \mapsto (1, n+1)$. Define $+, \cdot$ on $\mathbb{Z}$.

Finally, $\mathbb{Q} := \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \approx$

where $(m_1, n_1) \approx (m_2, n_2)$ iff $m_1 n_2 = m_2 n_1$

(secretly, this is $\frac{m_1}{n_1} = \frac{m_2}{n_2}$)

Define $+, \cdot$ on $\mathbb{Q}$, and define $<$ by

$(m_1, n_1) < (m_2, n_2)$ iff $n_1 n_2 (n_1 m_2 - n_2 m_1) \in \mathbb{N}$

(secretly, this is $\frac{m_2}{n_2} - \frac{m_1}{n_1} = \frac{n_1 m_2 - n_2 m_1}{n_1 n_2} > 0$)

Going back to construction of $\mathbb{R}$: Two possible approaches

1. Dedekind cuts
2. Completion via set of Cauchy sequences.

To be continued.