Lecture 10 Inequalities (Part 3)

From last time, we had Young's inequality:

If $A, B \geq 0$, $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}.$$

Today: Part 1: Another (more standard) proof of Hölder's inequality

Part 2: Cauchy-Schwarz, arithmetic mean vs geometric means.

<u>Part 1</u> Let's give a new proof for Hölder's:

If $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for Riemann integrable f, g on $[0,1]$,

$$\left| \int_0^1 f(x) g(x)\, dx \right| \leq \|f\|_{L^p} \|g\|_{L^q} . \quad \left(\text{recall } \|f\|_{L^p} = \left(\int_0^1 |f(x)|^p dx \right)^{\frac{1}{p}} \right).$$

Observation: Without loss of generality, assume $f, g \geq 0$ on $[0,1]$.

Easier task: If $\int_0^1 f(x)^p\,dx$ and $\int_0^1 g(x)^q\,dx$ are under control,

can we show that $\int_0^1 f(x)g(x)\,dx$ is not too large?

Yes, use Young's! For every $x \in [0,1]$,

$$f(x)g(x) \leq \frac{f(x)^p}{p} + \frac{g(x)^q}{q}$$

$$\xRightarrow[\text{over } [0,1]]{\text{integrate}} \quad \int_0^1 f(x)g(x)\,dx \leq \frac{1}{p}\int_0^1 f(x)^p\,dx + \frac{1}{q}\int_0^1 g(x)^q\,dx$$

$$\text{i.e. } \int_0^1 f(x)g(x)\,dx \leq \frac{1}{p}\|f\|_{L^p}^p + \frac{1}{q}\|g\|_{L^q}^q \quad \text{———} \quad (\ast)$$

But we need a product $\|f\|_{L^p}\|g\|_{L^q}$ on RHS, not a sum.

Trick: Apply $(\ast)$ to λf and $\frac{1}{\lambda}g$ where $\lambda > 0$ is some constant to be determined. Then $\int_0^1 f(x)g(x)\,dx = \int_0^1 \lambda f(x)\frac{1}{\lambda}g(x)\,dx \leq \frac{1}{p}\lambda^p\|f\|_{L^p}^p + \frac{1}{q\lambda^q}\|g\|_{L^q}^q$

and the RHS is the smallest when $\lambda = \left(\frac{\|g\|_{L^q}^q}{\|f\|_{L^p}^p}\right)^{\frac{1}{p+q}}$. Then

RHS $= \|f\|_{L^p}\|g\|_{L^q}$, as desired.

Part 2: Cauchy-Schwarz, arithmetic means vs geometric means

Cauchy-Schwarz is a special case of Hölder's inequality:

(✱✱) $$\left| \int_0^1 f(x)\, \overline{g(x)}\, dx \right| \le \|f\|_{L^2} \|g\|_{L^2} \qquad \left(\text{note } \tfrac{1}{2}+\tfrac{1}{2}=1\right)$$

for all Riemann integrable f, g on $[0,1]$.

Let's try to give another (non-standard) proof without using Hölder's.

Ideas from this other proof are useful in other parts of mathematics.

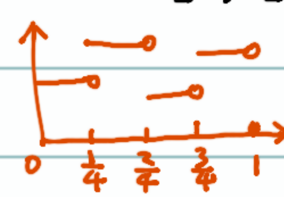
Idea: It suffices to prove Cauchy-Schwarz when f, g are constant on dyadic intervals of length 2^{-n} for an arbitrary $n \in \mathbb{N}$

i.e. for f, g that takes the form. $\sum_{m=1}^{2^n} a_m \chi_{\left[\frac{m-1}{2^n}, \frac{m}{2^n}\right)}(x)$

where $a_1, \cdots, a_{2^n}$ are constants.

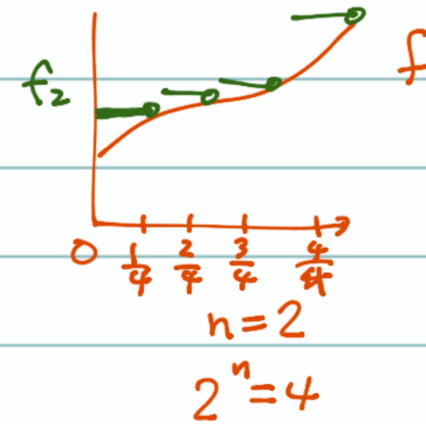
$\chi_{[a,b)}(x) = \begin{cases} 1 & \text{if } x \in [a,b) \\ 0 & \text{otherwise} \end{cases}$

eg. $n=2$: $2^n = 4$



This is because if f, g are Riemann integrable on $[0,1]$, then for each $n \in \mathbb{N}$, we can let

$$f_n(x) = \sum_{m=1}^{2^n} f\left(\frac{m}{2^n}\right) \chi_{\left[\frac{m-1}{2^n}, \frac{m}{2^n}\right)}(x)$$



$$g_n(x) = \sum_{m=1}^{2^n} g\left(\frac{m}{2^n}\right) \chi_{\left[\frac{m-1}{2^n}, \frac{m}{2^n}\right)}(x)$$

and

$$\left| \int_0^1 f(x)\overline{g(x)}\,dx \right| = \lim_{n\to\infty} \left| \int_0^1 f_n(x)\overline{g_n(x)}\,dx \right|$$

$$\int_0^1 |f(x)|^2 dx = \lim_{n\to\infty} \int_0^1 |f_n(x)|^2 dx$$

$$\int_0^1 |g(x)|^2 dx = \lim_{n\to\infty} \int_0^1 |g_n(x)|^2 dx .$$

Note if $f_n(x) = \sum_{m=1}^{2^n} a_m \chi_{[\frac{m-1}{2^n}, \frac{m}{2^n})}(x)$ and $g_n(x) = \sum_{m=1}^{2^n} b_m \chi_{[\frac{m-1}{2^n}, \frac{m}{2^n})}(x)$

then $\int_0^1 f_n(x)\overline{g_n(x)}\,dx = \frac{1}{2^n}\sum_{m=1}^{2^n} a_m \overline{b_m}$,

$$\|f_n\|_{L^2} = \left(\frac{1}{2^n}\sum_{m=1}^{2^n}|a_m|^2\right)^{1/2}, \quad \|g_n\|_{L^2} = \left(\frac{1}{2^n}\sum_{m=1}^{2^n}|b_m|^2\right)^{\frac{1}{2}}.$$

Thus our desired Cauchy-Schwarz inequality for functions follows, if we can prove that $\forall n \in \mathbb{N}$, $\forall a_1, \cdots, a_{2^n}, b_1, \cdots, b_{2^n} \in \mathbb{C}$, we have

$$\left|\sum_{m=1}^{2^n} a_m \overline{b_m}\right| \le \left(\sum_{m=1}^{2^n}|a_m|^2\right)^{\frac{1}{2}}\left(\sum_{m=1}^{2^n}|b_m|^2\right)^{1/2}$$

This can be proved by induction on n; since 2^{-n} is the "scale" at which f_n, g_n are constants, this argument is sometimes called induction on scales, which is useful in many other problems in analysis, PDE and number theory.

The induction argument goes as follows:

Let $P(n)$ be the proposition that

" $\left|\sum_{m=1}^{2^n} a_m \overline{b_m}\right| \le \left(\sum_{m=1}^{2^n} |a_m|^2\right)^{1/2} \left(\sum_{m=1}^{2^n} |b_m|^2\right)^{1/2} \quad \forall a_1, \dots, a_{2^n}, b_1, \dots, b_{2^n} \in \mathbb{C}$ ".

Base case $n=1$: we need to prove

$$|a_1\overline{b_1} + a_2\overline{b_2}|^2 \le (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2) \quad \forall a_1, a_2, b_1, b_2 \in \mathbb{C}$$

But this follows since

$$|a_1\overline{b_1} + a_2\overline{b_2}|^2 - (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2)$$

$$= 2\,\mathrm{Re}(a_1\overline{b_1}\,\overline{a_2}\,b_2) - |a_1|^2|b_2|^2 - |a_2|^2|b_1|^2$$

$$= -|a_1 b_2 - a_2 b_1|^2 \le 0.$$

Now suppose $k \in \mathbb{N}$ and $P(n)$ holds for all $n = 1, 2, \cdots, k$.

Want to show that $P(k+1)$ is true.

Let $a_1, a_2, \ldots, a_{2^{n+1}}, b_1, b_2, \ldots, b_{2^{n+1}} \in \mathbb{C}$.

Then $|a_1 \overline{b_1} + \cdots + a_{2^{n+1}} \overline{b_{2^{n+1}}}|$

$$\leq |a_1 \overline{b_1} + \cdots + a_{2^n} \overline{b_{2^n}}| + |a_{2^n+1} \overline{b_{2^n+1}} + \cdots + a_{2^{n+1}} \overline{b_{2^{n+1}}}|$$

$$\underset{\text{by } P(n)}{\leq} \left(|a_1|^2 + \cdots + |a_{2^n}|^2\right)^{1/2} \left(|b_1|^2 + \cdots + |b_{2^n}|^2\right)^{1/2} + \left(|a_{2^n+1}|^2 + \cdots + |a_{2^{n+1}}|^2\right)^{1/2} \left(|b_{2^n+1}|^2 + \cdots + |b_{2^{n+1}}|^2\right)^{1/2}$$

$$\underset{\text{by } P(1)}{\leq} \left(|a_1|^2 + \cdots + |a_{2^{n+1}}|^2\right)^{1/2} \left(|b_1|^2 + \cdots + |b_{2^{n+1}}|^2\right)^{1/2}$$

So $P(k+1)$ is true. This completes our alternative proof of Cauchy-Schwarz.

We remark that this also gives Cauchy-Schwarz in $\mathbb{C}^N$ $\forall N \in \mathbb{N}$:

via backward induction, we obtain

$$\left|\sum_{m=1}^{N} a_m \overline{b_m}\right| \le \left(\sum_{m=1}^{N} |a_m|^2\right)^{1/2} \left(\sum_{m=1}^{N} |b_m|^2\right)^{1/2} \quad \forall N \in \mathbb{N} \text{ (not just } N \in 2^{\mathbb{N}}\text{)}$$

(Just choose $n \in \mathbb{N}$ such that $2^n \ge N$, and note that

$$\sum_{m=1}^{N} a_m \overline{b_m} = \sum_{m=1}^{2^n} a_m \overline{b_m}$$

if $\begin{cases} a_{N+1} = \cdots = a_{2^n} = 0 \\ b_{N+1} = \cdots = b_{2^n} = 0 \end{cases}$)

Finally we prove the inequality between arithmetic and geometric means (in short, AM-GM inequality):

If $x_1, x_2, \ldots, x_n \geq 0$ then $\dfrac{x_1 + \cdots + x_n}{n} \geq (x_1 \cdots x_n)^{\frac{1}{n}}$. —— (***)

(***) is trivial for $n=1$. To prove (***) for $n=2$, let $x_1, x_2 \geq 0$.

Then

$$1 + \frac{\sqrt{x_1x_2}}{1!} + \frac{(\sqrt{x_1x_2})^2}{2!} + \cdots + \frac{(\sqrt{x_1x_2})^N}{N!}$$

$$= \sqrt{1}\sqrt{1} + \sqrt{\frac{x_1}{1!}}\sqrt{\frac{x_2}{1!}} + \sqrt{\frac{x_1^2}{2!}}\sqrt{\frac{x_2^2}{2!}} + \cdots + \sqrt{\frac{x_1^N}{N!}}\sqrt{\frac{x_2^N}{N!}}$$

$$\leq \left(1 + \frac{x_1}{1!} + \frac{x_1^2}{2!} + \cdots + \frac{x_1^N}{N!}\right)^{1/2}\left(1 + \frac{x_2}{1!} + \frac{x_2^2}{2!} + \cdots + \frac{x_2^N}{N!}\right)^{1/2} \quad \text{(Cauchy-Schwarz)}$$

so letting $N \to \infty$, we get $e^{\sqrt{x_1x_2}} \leq e^{\frac{x_1+x_2}{2}}$, i.e. $\sqrt{x_1x_2} \leq \dfrac{x_1+x_2}{2}$

From (***) for $n=2$ we can prove (***) for $n=2^k$ $\forall k \in \mathbb{N}$

(eg. $$\frac{x_1+x_2+x_3+x_4}{4} = \frac{1}{2}\left(\frac{x_1+x_2}{2} + \frac{x_3+x_4}{2}\right)$$
$$\geqslant \sqrt{\frac{x_1+x_2}{2}}\sqrt{\frac{x_3+x_4}{2}}$$
$$\geqslant (x_1 x_2 x_3 x_4)^{\frac{1}{4}}$$ via 2 applications of (***) for $n=2$)

A backward induction then proves (***) $\forall n \in \mathbb{N}$.

(eg. If (***) holds for $n=4$, then given $x_1, x_2, x_3 \geqslant 0$,
we let $x_4 = (x_1 x_2 x_3)^{\frac{1}{3}}$, and apply (***) to x_1, x_2, x_3, x_4:
then $$\frac{x_1+x_2+x_3+(x_1x_2x_3)^{\frac{1}{3}}}{4} \geqslant \left[x_1 x_2 x_3 (x_1x_2x_3)^{\frac{1}{3}}\right]^{\frac{1}{4}} = (x_1x_2x_3)^{\frac{1}{3}}$$
so rearranging gives $$\frac{x_1+x_2+x_3}{3} \geqslant (x_1x_2x_3)^{\frac{1}{3}}.$$)