Analysis I Add on Lecture 11

Aside: Revisit the Cauchy-Schwarz inequality.

Last time: If $a_1, \cdots, a_N, b_1, \cdots, b_N \in \mathbb{C}$, then

$$\left| \sum_{i=1}^{N} a_i b_i \right| \le \left(\sum_{i=1}^{N} |a_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} |b_i|^2 \right)^{\frac{1}{2}}$$

Question: Can we let $N \to \infty$?

Induction on N would only prove this statement $\forall N \in \mathbb{N}$.

but not for $N = \infty$.

Instead, need to pass to limit as $N \to \infty$.

<u>Theorem</u> Suppose $\{a_i\}_{i=1}^{\infty}$, $\{b_i\}_{i=1}^{\infty}$ are sequences of complex numbers such that

$$\sum_{i=1}^{\infty} |a_i|^2 < \infty, \qquad \sum_{i=1}^{\infty} |b_i|^2 < \infty$$

Then $$\sum_{i=1}^{\infty} a_i b_i := \lim_{N\to\infty} \sum_{i=1}^{N} a_i b_i \text{ exists in } \mathbb{C}$$

and $$\left| \sum_{i=1}^{\infty} a_i b_i \right| \le \left(\sum_{i=1}^{\infty} |a_i|^2 \right)^{1/2} \left(\sum_{i=1}^{\infty} |b_i|^2 \right)^{1/2}.$$

<u>Proof</u>. $$\left| \sum_{i=M}^{N} a_i b_i \right| \le \left(\sum_{i=M}^{N} |a_i|^2 \right)^{1/2} \left(\sum_{i=M}^{N} |b_i|^2 \right)^{1/2}$$

which is small as long as N, M are both large. So $\lim_{N\to\infty} \sum_{i=1}^{N} a_i b_i$ exists in $\mathbb{C}$, and now let $N\to\infty$ in

$$\left| \sum_{i=1}^{N} a_i b_i \right| \le \left(\sum_{i=1}^{N} |a_i|^2 \right)^{1/2} \left(\sum_{i=1}^{N} |b_i|^2 \right)^{1/2}$$

Today: Construction of the exponential function
(we used $\exp(x)$ in studying the Hölder's inequality and AM-GM inequality, so we might as well construct it)

<u>Definition</u> For $x \in \mathbb{R}$, define $\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

First, note that this series converges absolutely and uniformly on any compact subsets of $\mathbb{R}$.

Tool: Weierstrass M-test

Useful for proving uniform convergence
(i.e. convergence in the metric space with sup norm)

Prop (Weierstrass)

If $\{f_n(x)\}$ is a sequence of functions on a set A and $\{a_n\}$ is a sequence of non-negative real numbers such that

① $|f_n(x)| \leq a_n \quad \forall n \in \mathbb{N}, \ \forall x \in A$

② $\sum_{n=1}^{\infty} a_n < \infty$

then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to some function $f(x)$ on A

<u>Proof</u>. If $M > N$

$$\sup_{x \in A} \left| \sum_{n=1}^{N} f_n(x) - \sum_{n=1}^{M} f_n(x) \right| = \sup_{x \in A} \left| \sum_{n=N+1}^{M} f_n(x) \right|$$

$$\leq \sum_{n=N+1}^{M} a_n \text{ is small if } \underset{\text{large}}{N, M} \text{ are}$$

Application. let $[-M, M]$ be a closed and bounded interval

Want: $S_N(x) := \sum_{n=0}^{N} \frac{x^n}{n!}$ Converge uniformly on $[-M, M]$

as $N \to \infty$.

We need only to bound $\left|\frac{x^n}{n!}\right|$ by a constant depending on n, $\forall x \in [-M, M]$

But $\left|\frac{x^n}{n!}\right| \le \frac{M^n}{n(n-1)(n-2)\cdots 1}$

think: M fixed
n large
$(n \gg M)$

$$\le \frac{M^n}{n(n-1)\cdots(2M+1)}$$

$$\le \frac{M^n}{(2M)^{n-2M}} = (2M)^{2M}\left(\frac{1}{2}\right)^n$$

and $\sum_{n=0}^{\infty} (2M)^{2M}\left(\frac{1}{2}\right)^n < \infty$. So $S_N(x)$ converges uniformly on $[-M, M]$.

As a result, $\exp(x) = \lim_{N\to\infty} S_N(x)$ is a continuous function on $[-M, M]$ $\forall M > 0$, hence $\exp(x)$ is continuous on $\mathbb{R}$.

Furthermore, $S_N'(x) = \frac{d}{dx}\left(\sum_{n=0}^{N} \frac{x^n}{n!} \right) = \sum_{n=1}^{N} \frac{x^{n-1}}{(n-1)!} = S_{N-1}(x)$

so $S_N'(x)$ also converges uniformly to $\exp(x)$ on every compact subset of $\mathbb{R}$

As a result, $\exp(x)$ is differentiable on $\mathbb{R}$,

and $\frac{d}{dx}\exp(x) = \lim_{N\to\infty} S_N'(x) = \exp(x)$ $\forall x \in \mathbb{R}$.

We may now prove that

(i) $\exp(x+y) = \exp(x)\exp(y) \quad \forall x, y \in \mathbb{R}$

(ii) $\exp(x) > 0 \quad \forall x \in \mathbb{R}$

To see (i), fix $y \in \mathbb{R}$, consider $F(x) = \exp(x+y)\exp(-x)$

Then $F'(x) = \exp(x+y)\exp(-x) + \exp(x+y)[-\exp(-x)]$ (chain rule)

$= 0$

so $F(x) = F(0) = \exp(0+y)\exp(0) = \exp(y)$

i.e. $\exp(x+y)\exp(-x) = \exp(y) \quad \forall x, y \in \mathbb{R}$ ——— $(*)$

When $y=0$ this shows that $\exp(x)\exp(-x) = 1 \quad \forall x \in \mathbb{R}$

so multiplying both sides of $(*)$ by $\exp(x)$,

we get $\exp(x+y)\underbrace{\exp(-x)\exp(x)}_{=1} = \exp(x)\exp(y) \quad \forall x, y \in \mathbb{R}$

so $\exp(x+y) = \exp(x)\exp(y) \; \forall x, y \in \mathbb{R}$.

(ii) $\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots > 0$ if $x \geq 0$

Why is $\exp(x) > 0$ even if $x < 0$?

But then if $x < 0$,

then $\exp(x) = \frac{1}{\exp(-x)} > 0$ since then $\exp(-x) > 0$.

$\frac{d}{dx}\exp(x) > 0$

In particular, $\exp(x)$ is a strictly increasing function on $\mathbb{R}$

$\exp(x)$ is a strictly convex function on $\mathbb{R}$

$\frac{d^2}{dx^2}\exp(x) > 0$

Sometimes we want to know that

$$\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \qquad \forall x \in \mathbb{R}$$

To see this, Consider 2 cases: $x \geq 0$, $x < 0$.

Case 1 $x \geq 0$.

$$\left(1+\frac{x}{n}\right)^n = 1 + \frac{\cancel{n}}{1!}\left(\frac{x}{\cancel{n}}\right) + \frac{n(n-1)}{2!}\left(\frac{x}{n}\right)^2 + \cdots + \frac{n(n-1)\cdots 1}{n!}\left(\frac{x}{n}\right)^n$$

$$= 1 + \frac{x}{1!} + \left(1-\frac{1}{n}\right)\frac{x^2}{2!} + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{x^3}{3!} + \cdots$$

$$+ \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)\frac{x^n}{n!}$$

As a result, $\left(1+\frac{x}{n}\right)^n \leq 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$

so $\limsup_{n\to\infty}\left(1+\frac{x}{n}\right)^n \leq \exp(x)$.

On the other hand, the same expression for $\left(1+\frac{x}{n}\right)^n$ shows that

$$\left(1+\frac{x}{n}\right)^n \geq 1+\frac{x}{1!}+\left(1-\frac{1}{n}\right)\frac{x^2}{2!}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{x^3}{3!}+\cdots$$

$$+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)\frac{x^m}{m!}$$

where $m \leq n$.

Fix m, let $n \to \infty$.

$$\liminf_{n\to\infty}\left(1+\frac{x}{n}\right)^n \geq \liminf_{n\to\infty}\left[1+\frac{x}{1!}+\left(1-\frac{1}{n}\right)\frac{x^2}{2!}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{x^3}{3!}\right.$$

$$\left.+\cdots+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)\frac{x^m}{m!}\right]$$

$$= 1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^m}{m!}$$

But this is true $\forall m$: so let $m \to \infty$, $\liminf_{n\to\infty}\left(1+\frac{x}{n}\right)^n \geq \exp(x)$

When $x \geqslant 0$, we saw

$$\exp(x) \leqslant \liminf_{n\to\infty}\left(1+\frac{x}{n}\right)^n \leqslant \limsup_{n\to\infty}\left(1+\frac{x}{n}\right)^n \leqslant \exp(x)$$

hence $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ exists and equals $\exp(x)$.

Case $x<0$: let $y=-x>0$.

Want to prove: $\lim_{n\to\infty}\left(1-\frac{y}{n}\right)^n = \exp(-y)$

But we knew $\exp(-y) = \dfrac{1}{\exp(y)} = \dfrac{1}{\lim_{n\to\infty}\left(1+\frac{y}{n}\right)^n}$

and $\left(1-\frac{y}{n}\right)^n = \dfrac{\left[1-\left(\frac{y}{n}\right)^2\right]^n}{\left(1+\frac{y}{n}\right)^n}$

So it suffices to prove that $\lim_{n\to\infty}\left(1-\frac{y^2}{n^2}\right)^n = 1 \ \forall y>0.$

<u>Lemma</u> If $0 \le t \le 1$ and $n \in \mathbb{N}$, then

$$1 - nt \le (1-t)^n \le 1$$

<u>Proof</u> Clearly $(1-t)^n \le 1$. We prove $(1-t)^n \ge 1 - nt$ by induction on n:

$n = 1 \Rightarrow (1-t)^n = 1 - t = 1 - nt$

If $(1-t)^k \ge 1 - kt$ for some positive integer k, then

$$\begin{aligned} (1-t)^{k+1} &= (1-t)^k (1-t) \\ &\ge (1-kt)(1-t) \\ &= 1 - (k+1)t + kt^2 \\ &\ge 1 - (k+1)t . \end{aligned}$$

$\blacksquare$

Using Lemma, $\forall y > 0$, we have

$$1 - n\left(\frac{y^2}{n^2}\right) \le \left(1 - \frac{y^2}{n^2}\right)^n \le 1 \qquad \forall n \ge y$$

Let $n \to \infty$: $\lim_{n\to\infty}\left(1 - \frac{y^2}{n^2}\right)^n = 1$, as desired.