Analysis 1 Add on Lecture 12

Last time: We defined the exponential function $\exp : \mathbb{R} \to (0,\infty)$

by $\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \quad \forall x \in \mathbb{R}$

We proved that $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \exp(x)$ if $x \geqslant 0$

and we now want to prove that this also holds if $x < 0$.

Let $y = -x > 0$ Want to prove $\lim_{n\to\infty} \left(1 - \frac{y}{n}\right)^n = \exp(-y)$

But we knew already that $\exp(-y) = \frac{1}{\exp(y)} = \frac{1}{\lim_{n\to\infty} \left(1 + \frac{y}{n}\right)^n}$

Goal: $\displaystyle\lim_{n\to\infty}\left(1-\frac{y}{n}\right)^n = \frac{1}{\displaystyle\lim_{n\to\infty}\left(1+\frac{y}{n}\right)^n}$ if $y>0$.

But $\displaystyle\left(1-\frac{y}{n}\right)^n = \frac{\left(1-\frac{y^2}{n^2}\right)^n}{\left(1+\frac{y}{n}\right)^n}$, so it suffices to show that

$$\lim_{n\to\infty}\left(1-\boxed{\frac{y^2}{n^2}}\right)^n = 1.$$

think: try to understand $(1-t)^n$ is when t is extremely small.

Lemma. For $0\le t\le 1$, $n\in\mathbb{N}$, we have $1-nt \le (1-t)^n \le 1$.

Assuming the lemma, we have, if $y>0$, and $n\ge y$, then $1-\cancel{n}\cdot\frac{y^2}{n^{\cancel{2}}} \le \left(1-\frac{y^2}{n^2}\right)^n \le 1$

so $\lim_{n\to\infty}\left(1-\frac{y^2}{n^2}\right)^n = 1$ by sandwich theorem.

<u>Proof of lemma</u> For $0\leq t\leq 1$, $n\in\mathbb{N}$, want to prove that $(1-t)^n \geq 1-nt$.

<u>Method 1</u> Inducting on n.

<u>Method 2</u>. Differentiation. Consider $f(t) = (1-t)^n - (1-nt)$ and show that

$$\begin{cases} f(0)=0 \\ f'(t)\geq 0 \quad \forall t\in(0,1) \end{cases} \quad \text{then} \quad f(t)\geq 0 \quad \forall t\in[0,1].$$

<u>Method 3</u> Integration. $1-(1-t)^n = \int_{1-t}^{1} ns^{n-1}\,ds \leq n\int_{1-t}^{1} ds = nt$

Method 4 Binomial theorem

$$1 = [(1-t)+t]^n = (1-t)^n + n(1-t)^{n-1}t + \frac{n(n-1)}{2!}(1-t)^{n-2}t^2 + \cdots + \frac{n(n-1)\cdots 1}{n!}t^n$$

$$= (1-t)^n + nt\left[(1-t)^{n-1} + \frac{n-1}{2!}(1-t)^{n-2}t + \cdots + \frac{(n-1)\cdots 1}{n!}t^{n-1}\right]$$

$$\leq (1-t)^n + nt\left[(1-t)^{n-1} + \frac{n-1}{1!}(1-t)^{n-2}t + \cdots + \frac{(n-1)\cdots 1}{(n-1)!}t^{n-1}\right]$$

$$= (1-t)^n + nt\left[(1-t)+t\right]^{n-1}$$

$$= (1-t)^n + nt.$$

Earlier in this course: Fourier series, inequality between arithmetic and geometric means.

Today: Application towards isoperimetric inequality in 2-dimensions.

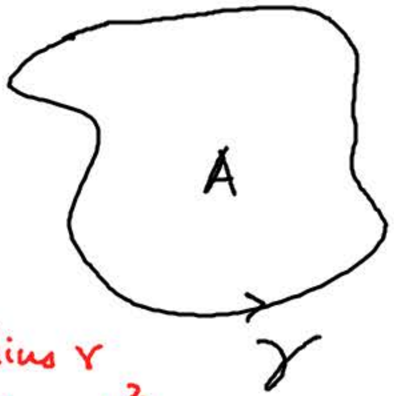
<u>Theorem</u> Let γ be a smooth, simple closed curve in $\mathbb{R}^2$

Let L be the length of γ.

A be the area enclosed by γ

Then $A \leq \dfrac{L^2}{4\pi}$, with equality iff γ is a circle.

$L = |\gamma|$



eg. γ = circle of radius r

$L = 2\pi r$, $A = \pi r^2 = \dfrac{L^2}{4\pi}$

<u>Proof</u> Let γ be a smooth, simple closed curve

Then γ can be parametrized by $(x(s), y(s))$ for $s \in [0,1]$

where $x(s), y(s)$ are real-valued C^∞ functions with $\begin{cases} x(0)=x(1) \\ y(0)=y(1) \end{cases}$

and $(x(s),y(s)) \neq (x(t), y(t))$ if $0 \leq s < t < 1$; we may also

extend $x(s), y(s)$ to be C^∞ functions on $\mathbb{R}$ that are periodic of

period 1. Fourier expansion gives

$$x(s) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n s}, \quad y(s) = \sum_{n=-\infty}^{\infty} b_n e^{2\pi i n s}$$

where $a_n, b_n \in \mathbb{C}$ $\forall n \in \mathbb{Z}$.

But $A = \left|\int_0^1 x(s)\,\overline{y'(s)}\,ds\right|$

$L = \left(\int_0^1 x'(s)^2 + y'(s)^2\,ds\right)^{1/2}$

so $A = \left|\sum_{n=-\infty}^{\infty} a_n \cdot \overline{2\pi i n b_n}\right|$

$= 2\pi \left|\sum_{n=-\infty}^{\infty} n a_n \overline{b_n}\right|$

$\le 2\pi \sum_{n\in\mathbb{Z}\setminus\{0\}} |n \underline{\underline{a_n b_n}}|$

$$L^2 = \sum_{n=-\infty}^{\infty}\left(|2\pi i n a_n|^2 + |2\pi i n b_n|^2\right) = 4\pi^2 \sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^2\left(\underline{\underline{|a_n|^2 + |b_n|^2}}\right).$$

$A = \left|\int_\gamma x\,dy\right|$ (because the integral is $\int_\Omega dxdy = A$)

$x(s) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n s}$

$y(s) = \sum_{n=-\infty}^{\infty} b_n e^{2\pi i n s}$

As a result,

$$A \leq 2\pi \sum_{n \in \mathbb{Z} \setminus \{0\}} |n|\, |a_n b_n|$$

$$\leq 2\pi \sum_{n \in \mathbb{Z} \setminus \{0\}} |n| \frac{|a_n|^2 + |b_n|^2}{2} \qquad \text{(AM-GM)}$$

$$\leq 2\pi \sum_{n \in \mathbb{Z} \setminus \{0\}} |n|^2 \frac{|a_n|^2 + |b_n|^2}{2}$$

$|n| \leq |n|^2$ if n is a non-zero integer.
equality holds iff $n = \pm 1$

$$= 2\pi \frac{L^2}{4\pi^2} = \frac{L^2}{4\pi},$$

with equality iff $a_n = b_n = 0$ $\forall n \in \mathbb{N}$ with $|n| \geq 2$.

Equality happens iff γ is a circle.