Analysis I Add on Lecture 2

Last time: Construction of $(\mathbb{Q}, +, \cdot, <)$

Today: Construct the real number system $(\mathbb{R}, +, \cdot, <)$

(Part 1)

Two approaches:

① Dedekind cuts

Key observation: If x is a real number, then it "cuts" the set of rationals into two halves, namely $\{r \in \mathbb{Q} : r < x\}$ and $\{r \in \mathbb{Q} : r \geq x\}$.

Conversely, consider a set of rationals S with the following properties:

(i) $S \neq \phi$, $\mathbb{Q} \setminus S \neq \phi$

(ii) If $r \in S$, then $r' \in S$ $\forall r' \in \mathbb{Q}$, with $r' < r$.

(iii) S does not contain a largest element

i.e. if $r \in S$, then $\exists r'' \in S$ with $r'' > r$.

Note that if secretly $x \in \mathbb{R}$, then the set $\{r \in \mathbb{Q} : r < x\}$ satisfies these 3 properties. And we can turn this around: if S is such a set of rationals, then $S = \{r \in \mathbb{Q} : r < x\}$ for some real number x.

The set of all such sets S of rationals is in one-to-one correspondence with $\mathbb{R}$, namely $\{S\} \longleftrightarrow \mathbb{R}$.

The latter is well-defined using only what we know of $\mathbb{Q}$, and hence we can <u>define</u> $\mathbb{R}$ as the set of all such sets S. Then $\mathbb{Q} \to \{S\}$ where $r \in \mathbb{Q} \mapsto \{r' \in \mathbb{Q} : r' < r\}$ is an embedding of $\mathbb{Q}$ into $\mathbb{R}$.

② Completion via Cauchy sequences.

A sequence of rationals $(r_n)_{n\in\mathbb{N}}$ is said to be Cauchy, if $\forall \varepsilon \in \mathbb{Q}_+$ (:= set of positive rationals), $\exists N \in \mathbb{N}$ such that whenever $m, n \geq N$, we have

$$|r_n - r_m| < \varepsilon$$

<u>Define</u> $\mathbb{R}$:= set of all Cauchy sequences of rationals$/\sim$ where $(r_n)_{n\in\mathbb{N}} \sim (s_n)_{n\in\mathbb{N}}$ iff $\forall \varepsilon \in \mathbb{Q}_+$, $\exists N \in \mathbb{N}$ such that $|r_n - s_n| < \varepsilon \quad \forall n \geq N$.

(secretly, just $\lim_{n\to\infty}(r_n - s_n) = 0$.)

The map $\mathbb{Q} \longmapsto \{(r_n)\}$ given by $r \in \mathbb{Q} \mapsto (r, r, r, \cdots)$ is an embedding of $\mathbb{Q}$ into $\mathbb{R}$.

(Secretly, we want to say that a real number is a limit of some Cauchy sequence of rationals. But we can't talk about limits of Cauchy sequences before we have the reals. So we say a real number <u>is</u> a Cauchy sequence of rationals (rather than its limit) and work with this definition)

Example

(i) The sequence $(2^{-n})_{n\in\mathbb{N}}$ is a Cauchy sequence of rationals.

Proof. Let $\varepsilon \in \mathbb{Q}_+$, say $\varepsilon = \frac{a}{b}$ where $a, b \in \mathbb{N}$.

Want $N \in \mathbb{N}$ (possibly depending on a, b) such that $|2^{-n} - 2^{-m}| < \frac{a}{b}$ whenever $n, m \geq N$.

We may just take $N = b$, because if $n, m \geq b$, say $n \geq m$, then $|2^{-n} - 2^{-m}| = 2^{-m} - 2^{-n}$

$$< 2^{-m} \leq 2^{-b} \leq \frac{1}{b} \leq \frac{a}{b} = \varepsilon.$$

$$\left(\text{Reason: } 2^b = (1+1)^b = 1 + b + \binom{b}{2} + \cdots + \binom{b}{b} \geq b\right)$$

This shows $(2^{-n})_{n\in\mathbb{N}}$ is Cauchy.

(ii) $(2^{-n})_{n\in\mathbb{N}} \sim (0, 0, 0, \cdots)$; "same" proof as above.

(secretly: $\lim_{n\to\infty} 2^{-n} = 0$).

More details about constructing $\mathbb{R}$ via Cauchy sequences of rationals.

<u>Define</u> $\mathbb{R} := \{\text{Cauchy sequences of rationals}\} / \sim$ (So from now on, a real number is by definition a Cauchy sequence of rational numbers (r_n), often written $(r_n) \in \mathbb{R}$, up to the equivalence relation $\sim$.)

(a) Show that $\sim$ is an equivalent relation.

(i) Reflexivity: If $(r_n) \in \mathbb{R}$ then $(r_n) \sim (r_n)$. (Secretly: $\lim_{n\to\infty}(r_n - r_n) = 0$).

(ii) Symmetry: If $(r_n) \sim (s_n)$, then $(s_n) \sim (r_n)$.

(Secretly: If $\lim_{n\to\infty}(r_n - s_n) = 0$, then $\lim_{n\to\infty}(s_n - r_n) = 0$.)

(iii) Transitivity: If $(r_n) \sim (s_n)$ and $(s_n) \sim (t_n)$, then $(r_n) \sim (t_n)$.

(Secretly: If $\lim_{n\to\infty}(r_n - s_n) = 0$ and $\lim_{n\to\infty}(s_n - t_n) = 0$, then $\lim_{n\to\infty}(r_n - t_n) = 0$.)

(b) Define addition $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ via

$$(r_n) + (s_n) := (r_n + s_n). \qquad \forall (r_n), (s_n) \in \mathbb{R}$$

and define multiplication $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$(r_n) \cdot (s_n) := (r_n s_n) \qquad \forall (r_n), (s_n) \in \mathbb{R}$$

This is well-defined: because

(i) If $(r_n), (s_n)$ are Cauchy sequences of rationals, then so is $(r_n + s_n)$, and $(r_n s_n)$.

(Hint: $|r_n s_n - r_m s_m|$

$$\leq |r_n (s_n - s_m)| + |(r_n - r_m) s_m|$$

& Cauchy sequences are bounded ...)

(ii) If $(r_n') \sim (r_n)$ and $(s_n') \sim (s_n)$, then $(r_n' + s_n') \sim (r_n + s_n)$ and $(r_n' s_n') \sim (r_n s_n)$.

(Exercise).

(c) Show that $(\mathbb{R}, +)$ is an abelian group, with additive identity $0 := (0, 0, \cdots)$, and $(\mathbb{R} \setminus \{0\}, \cdot)$ is an abelian group; also distribution law holds:

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \forall a, b, c \in \mathbb{R}$$

Hence $(\mathbb{R}, +, \cdot)$ is a field.

These just follow from the corresponding properties of $(\mathbb{Q}, +, \cdot)$

To be continued...