Analysis I Add on Lecture 3.

Last time: Construction of real number system $(\mathbb{R},+,\cdot,<)$
from $(\mathbb{Q},+,\cdot,<)$

$$\mathbb{R} := \{\text{Cauchy sequences of rationals}\}/\sim$$
$$= \{(a_n) : (a_n) \text{ is a Cauchy sequence of rationals}\}/\sim$$

where $(a_n) \sim (b_n)$ iff $\forall \varepsilon \in \mathbb{Q}_+, \exists N \in \mathbb{N}$ such that
$|a_n - b_n| < \varepsilon \quad \forall n \geq N.$

Define $\begin{cases} (a_n) + (b_n) = (a_n + b_n) & \forall (a_n), (b_n) \in \mathbb{R} \\ (a_n) \cdot (b_n) = (a_n \cdot b_n) & \text{(well-defined!)} \end{cases}$

Also easy to check that $(\mathbb{R}, +, \cdot)$ is a field,
with additive identity being $(0, 0, 0, \cdots)$

$\mathbb{Q}$ embeds into $\mathbb{R}$ via

$$\underset{\mathbb{Q}}{\underset{\cap}{r}} \mapsto \underset{\mathbb{R}}{\underset{\cap}{(r, r, r, \cdots)}}$$

meaning that $(\mathbb{Q}, +, \cdot) \longrightarrow (\mathbb{R}, +, \cdot)$ is an injective field homomorphism.

Today: Construct $<$ on $\mathbb{R}$ that is compatible with the field structure of $\mathbb{R}$, and the $<$ on $\mathbb{Q}$.

<u>Definition</u>. Given $(a_n), (b_n) \in \mathbb{R}$, we say $(a_n) < (b_n)$ iff $\exists\, \varepsilon \in \mathbb{Q}_+$ and $N \in \mathbb{N}$ such that
$$b_n - a_n > \varepsilon \quad \forall n \geq N.$$

Let's check that this is well-defined:
If $(a_n) \sim (a_n')$ and $(b_n) \sim (b_n')$, when $(a_n) < (b_n)$ we want to show that $(a_n') < (b_n')$. But since $(a_n) < (b_n)$, $\exists\, \varepsilon \in \mathbb{Q}_+$ and $N \in \mathbb{N}$ such that $b_n - a_n > \varepsilon$ $\forall n \geq N$. Now take $M \in \mathbb{N}$ such that $\begin{cases} |a_n - a_n'| < \frac{\varepsilon}{3} & \forall n \geq M \\ |b_n - b_n'| < \frac{\varepsilon}{3} & \forall n \geq M. \end{cases}$

Then whenever $n \geq \max\{N, M\}$, we have
$$\begin{aligned} b_n' - a_n' &= (b_n' - b_n) + (b_n - a_n) + (a_n - a_n') \\ &> -\frac{\varepsilon}{3} + \varepsilon - \frac{\varepsilon}{3} = \frac{\varepsilon}{3}. \end{aligned}$$
Hence $(a_n') < (b_n')$ $\blacksquare$.

Properties of $(\mathbb{R}, +, \cdot, <)$

① . Trichotomy : Given any $(a_n), (b_n) \in \mathbb{R}$, exactly one of the following must hold:

$$(a_n) < (b_n), \quad (a_n) \sim (b_n), \quad (b_n) < (a_n).$$

② Transitivity: If $(a_n), (b_n), (c_n) \in \mathbb{R}$ and $(a_n) < (b_n)$, $(b_n) < (c_n)$, then $(a_n) < (c_n)$.

③ Compatibility with addition: If $(a_n), (b_n) \in \mathbb{R}$, and $(0) < (a_n)$, $(0) < (b_n)$, then $(0) < (a_n) + (b_n)$.

④ Compability with multiplication: If $(a_n), (b_n) \in \mathbb{R}$, and $(0) < (a_n)$, $(0) < (b_n)$, then $(0) < (a_n) \cdot (b_n)$.

<u>Notations</u>

We also write $(a_n) \leq (b_n)$ if $\big((a_n) < (b_n)$ or $(a_n) \sim (b_n) \big)$

and write $(a_n) > (b_n)$ if $(b_n) < (a_n)$.

Proofs ④ Suppose $(a_n) > (0)$ and $(b_n) > (0)$.

Want to show $(a_n) \cdot (b_n) > (0)$

i.e. $(a_n \cdot b_n) > (0)$.

By assumption, $\exists \varepsilon_1 \in \mathbb{Q}_+$, $N_1 \in \mathbb{N}$ such that

$$a_n - 0 > \varepsilon_1 \quad \forall n \geq N_1$$

Also, $\exists \varepsilon_2 \in \mathbb{Q}_+$, $N_2 \in \mathbb{N}$ such that

$$b_n - 0 > \varepsilon_2 \quad \forall n \geq N_2.$$

Then whenever $n \geq \max\{N_1, N_2\}$,

we have $a_n \cdot b_n - 0 > \varepsilon_1 \varepsilon_2 \in \mathbb{Q}_+$.

So $(a_n \cdot b_n) > (0)$, as desired.

Proof of ③ similar, proof of ② easy.

① : Let $(a_n), (b_n) \in \mathbb{R}$. Clearly we cannot have both $(a_n) < (b_n)$ and $(b_n) < (a_n)$. So at most one of them happens, and if none of them happens, We want to show that $(a_n) \sim (b_n)$.

Given $\varepsilon \in \mathbb{Q}_+$, choose $N \in \mathbb{N}$ such that

$$\begin{cases} |a_n - a_m| < \frac{\varepsilon}{3} \\ |b_n - b_m| < \frac{\varepsilon}{3} \end{cases} \quad \forall n, m \geq N.$$

(Possible since $(a_n), (b_n)$ are Cauchy).

Since $(a_n) \not< (b_n)$, there exists $m_1 \geq N$ such that $b_{m_1} - a_{m_1} \leq \frac{\varepsilon}{3}$. Hence $\forall n \geq N$,

$$b_n - a_n = (b_n - b_{m_1}) + (b_{m_1} - a_{m_1}) + (a_{m_1} - a_n)$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Similarly, since $(b_n) \not< (a_n)$, there exists $m_2 \geq N$ such that $a_{m_2} - b_{m_2} \leq \frac{\varepsilon}{3}$. Hence $\forall n \geq N$,

$$a_n - b_n = (a_n - a_{m_2}) + (a_{m_2} - b_{m_2}) + (b_{m_2} - b_n)$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence $|b_n - a_n| < \varepsilon \quad \forall n \geq N$. It follows that $(a_n) \sim (b_n)$.

We also need to show that if $(a_n) < (b_n)$ or if $(b_n) < (a_n)$ then $(a_n) \not\sim (b_n)$.
This is obvious from definition.

Remark. If $r, s \in \mathbb{Q}$, then $r < s$ iff $(r, r, r, \dots) < (s, s, s, \dots)$ so the $<$ on $\mathbb{R}$ is an extension of that on $\mathbb{Q}$.

Summary: We have now constructed an ordered field $(\mathbb{R}, +, \cdot, <)$ that extends $(\mathbb{Q}, +, \cdot, <)$.

Next time: State and prove the Completeness axiom of $\mathbb{R}$.

To be Continued...