Analysis I Add on Lecture 4

Previous 2 Lectures: Construction of an ordered field $(\mathbb{R}, +, \cdot, <)$ from $(\mathbb{Q}, +, \cdot, <)$

$\mathbb{R} := \{\text{Cauchy sequences of rationals}\} / \sim$, where

$(r_n) \sim (s_n)$ iff $\forall \varepsilon \in \mathbb{Q}_+, \exists N \in \mathbb{N}$ s.t. $|r_n - s_n| < \varepsilon \ \forall n \geq N$

" $\lim_{n\to\infty} (r_n - s_n) = 0$ ".

$(r_n) + (s_n) := (r_n + s_n)$; $(r_n) \cdot (s_n) := (r_n \cdot s_n)$.

$(r_n) < (s_n)$ iff $\exists \varepsilon \in \mathbb{Q}_+ \ \exists N \in \mathbb{N}$ s.t. $s_n - r_n > \varepsilon \ \forall n \geq N$;

write $(r_n) \leq (s_n)$ for " $(r_n) < (s_n)$ or $(r_n) \sim (s_n)$ ".

$\mathbb{Q}$ embeds into $\mathbb{R}$ via $r \in \mathbb{Q} \mapsto (r, r, r, \dots) \in \mathbb{R}$. (injective order-preserving field homomorphism)

This time: Completeness of $\mathbb{R}$ via the least upper bound axiom

Definition ① $(r_n) \in \mathbb{R}$ is said to be an upper bound for a subset S of $\mathbb{R}$, if

$$(s_n) \leq (r_n) \quad \forall (s_n) \in S$$

② $(r_n) \in \mathbb{R}$ is said to be the least upper bound of S, if (r_n) is an upper bound for S, and any $(r_n') < (r_n)$ is not an upper bound for S.

Theorem (Completeness axiom)

If $S \subseteq \mathbb{R}$ is non-empty and has an upper bound, then S has the least upper bound.

Lemma. If $S \subseteq \mathbb{R}$ has an upper bound, then $\exists u \in \mathbb{Q}$ such that u is an upper bound for S.

Proof Let (r_n) be an upper bound for S. Then (r_n) is Cauchy, and hence bounded. Let $u \in \mathbb{Q}$ be such that $r_n \leq u$ $\forall n \in \mathbb{N}$. Then $(r_n) \leq (u, u, u, \ldots)$ (Why?) Hence u (i.e. $(u, u, u, \cdots)$) is an upper bound for S. ∎

Lemma. If $S \subseteq \mathbb{R}$ and $S \neq \phi$, then $\exists \ell \in \mathbb{Q}$ such that ℓ is not an upper bound for S.

Proof. Let $(s_n) \in S$. Then (s_n) is Cauchy and hence bounded. Let $\ell \in \mathbb{Q}$ such that $s_n \geq \ell + 1$ $\forall n \in \mathbb{N}$. Then $(s_n) > (\ell, \ell, \ell, \cdots)$, and ℓ is not an upper bound for S. ∎

Proof of the completeness axiom

We use the method of bisection.

Let $S \subseteq \mathbb{R}$, $S \neq \phi$, and suppose S has an upper bound.

By lemmas, we can choose $u_1 \in \mathbb{Q}$, $l_1 \in \mathbb{Q}$, such that

$$\begin{cases} u_1 \text{ is an upper bound for } S \\ l_1 \text{ is not an upper bound for } S. \end{cases}$$

If $\frac{u_1+l_1}{2}$ is an upper bound for S,

then $\begin{cases} u_2 := \frac{u_1+l_1}{2} \\ l_2 := l_1 \end{cases}$. Otherwise $\begin{cases} u_2 := u_1 \\ l_2 := \frac{u_1+l_1}{2} \end{cases}$

Then u_2 is an upper bound for S,

l_2 is not an upper bound for S,

and $l_1 \leq l_2 \leq u_2 \leq u_1$, $u_2 - l_2 = \frac{u_1 - l_1}{2}$.

Repeat, we get two sequences $(u_1, u_2, u_3, \cdots)$ and $(l_1, l_2, l_3, \cdots)$ of rationals, such that

$$\begin{cases} u_n \text{ is an upper bound for } S \\ l_n \text{ is not an upper bound for } S, \end{cases}$$

and $l_1 \leq l_2 \leq l_3 \leq \cdots \leq u_3 \leq u_2 \leq u_1$,

with $u_n - l_n = \dfrac{u_1 - l_1}{2^{n-1}}$ $\forall n \in \mathbb{N}$.

In particular, (u_n) and (l_n) are Cauchy sequences of rationals, and $(u_n) \sim (l_n)$.

(c.f. example in lecture 2:
We showed that $\left(\frac{1}{2^n}\right) \sim (0, 0, 0, \cdots)$).

<u>Claim 1</u>. (u_n) is an upper bound for S

<u>Claim 2</u>. If $(r_n) < (l_n)$, then (r_n) is not an upper bound for S.

<u>Proof of claim 1</u> Suppose (u_n) is not an upper bound for S, i.e. $\exists (s_n) \in S$ such that $(s_n) > (u_n)$.

Then $\exists \varepsilon \in \mathbb{Q}_+$ $\exists N \in \mathbb{N}$ such that $s_n - u_n > \varepsilon$ $\forall n \geq N$.

Choose $M \in \mathbb{N}$ such that $|u_n - u_m| < \frac{\varepsilon}{2}$ $\forall n, m \geq M$.

and take $K = \max\{M, N\}$.

Then $\forall n \geq K$, we have

$$\begin{aligned} s_n - u_K &= (s_n - u_n) + (u_n - u_K) \\ &> \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} \end{aligned}$$

which shows that $(s_n) > (u_K, u_K, u_K, \cdots)$

This is a contradiction since u_K is an upper bound for S.

$\blacksquare$

Proof of claim 2. Suppose $(r_n) < (\ell_n)$. Want to show that (r_n) is not an upper bound for S.

Since $(r_n) < (\ell_n)$, $\exists\, \varepsilon \in \mathbb{Q}_+\ \exists N \in \mathbb{N}$ such that

$\ell_n - r_n > \varepsilon \quad \forall n \geq N.$

Choose $M \in \mathbb{N}$ such that $|\ell_n - \ell_m| < \frac{\varepsilon}{2} \quad \forall n, m \geq M$.

and take $K = \max\{M, N\}$.

But ℓ_K is not an upper bound for S.

Hence $\exists\, (s_n) \in S$ such that $(\ell_K, \ell_K, \ell_K, \cdots) < (s_n)$.

As a result, $\exists\, \varepsilon' \in \mathbb{Q}_+, \exists N' \in \mathbb{N}$ such that

$s_n - \ell_K > \varepsilon' \quad \forall n \geq N'$.

Altogether, $\forall n \geq \max\{K, N'\}$, we have

$$\begin{aligned} s_n - r_n &= (s_n - \ell_K) + (\ell_K - \ell_n) + (\ell_n - r_n) \\ &> \varepsilon' - \frac{\varepsilon}{2} + \varepsilon = \frac{\varepsilon}{2} + \varepsilon', \text{ i.e. } (s_n) > (r_n) \end{aligned}$$

∎

This completes the proof of the completeness axiom.

And from here, you can prove

Corollary Every Cauchy sequence of real numbers has a limit in $\mathbb{R}$.

Corollary Every sequence of real numbers that is increasing and bounded above has a limit in $\mathbb{R}$.

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