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Analysis 1 Add on Lecture 5:

Completeness of metric spaces.

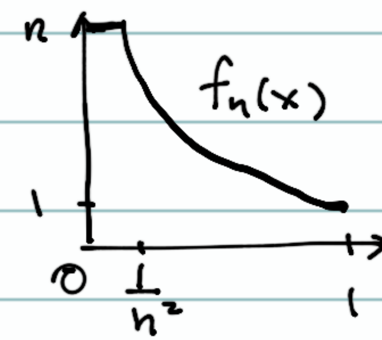
Examples.

① $C^0[0,1]$ = space of complex-valued continuous functions on $[0,1]$.

$\|f\|_{L^1} := \int_0^1 |f(x)|\,dx \qquad d(f,g) = \|f-g\|_{L^1}$

Then $(C^0[0,1], L^1)$ is not complete.

Reason: let $f_n(x) = \begin{cases} n & \text{if } 0 \le x \le \frac{1}{n^2} \\ \frac{1}{\sqrt{x}} & \text{if } \frac{1}{n^2} < x \le 1 . \end{cases}$

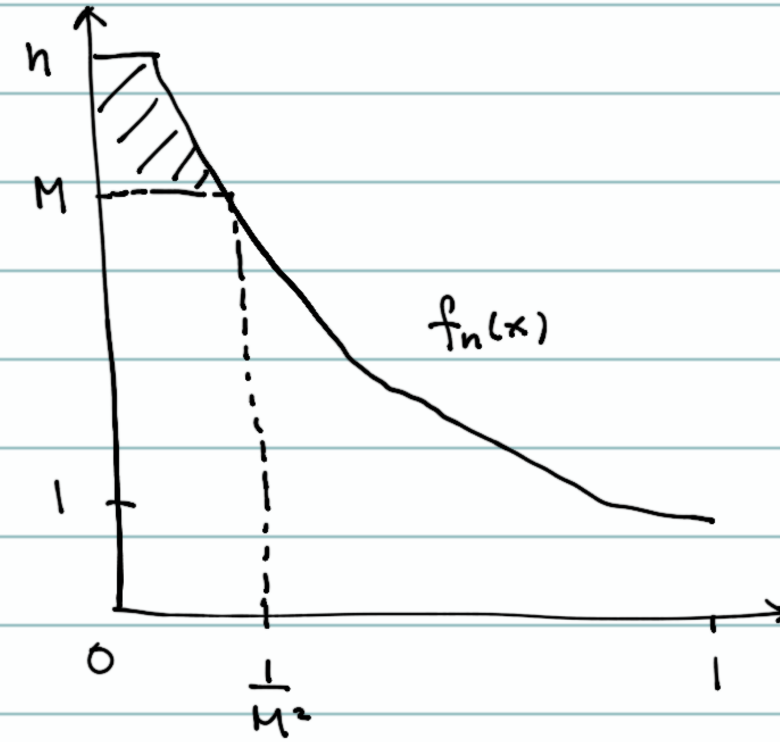


Easy to check that $\{f_n\}$ is a Cauchy sequence in $(C^0[0,1], L^1)$.

But if $f \in C^0[0,1]$, then $\|f_n - f\|_{L^1}$ does not tend to 0 as $n \to \infty$

Indeed, if $f \in C^0[0,1]$, then $\exists M$ such that $|f(x)| \le M$ $\forall x \in [0,1]$.

Then $\forall n > M$,



$$\int_0^1 |f_n - f|\,dx \ge \int_0^1 \mathrm{Re}(f_n - f)\,dx$$

$$\ge \int_{\frac{1}{(M+1)^2}}^{\frac{1}{M^2}} f_n(x) - \mathrm{Re} f(x)\,dx$$

$$\ge \underbrace{\int_{\frac{1}{(M+1)^2}}^{\frac{1}{M^2}} \frac{1}{\sqrt{x}} - M \quad dx}_{\text{fixed positive number independent of } n.}$$

(since $\frac{1}{\sqrt{x}} > M$ $\forall x \in \left(\frac{1}{(M+1)^2}, \frac{1}{M^2}\right)$.) So $\|f_n - f\|_{L^1} \not\to 0$ and $f \in C^0[0,1]$ is not the limit of $\{f_n\}$ in $(C^0[0,1], L^1)$.

② $(C^0[0,1], \|\cdot\|_{L^2})$ is not complete

$\left(\|f\|_{L^2} := \left(\int_0^1 |f(x)|^2 dx\right)^{\frac{1}{2}} \right)$

Proof. Exercise.

③. $\ell^1(\mathbb{Z}) = \{ (x_n)_{n=-\infty}^{\infty}$ sequence of complex numbers, $\sum_{n=-\infty}^{\infty} |x_n| < \infty \}$.

with $$\|(x_n)\|_{\ell^1} := \sum_{n=-\infty}^{\infty} |x_n| \left(= \lim_{N\to\infty} \sum_{n=-N}^{N} |x_n| \right)$$

Then $\ell^1(\mathbb{Z})$ is complete.

Proof. Let $(x^{(j)})_{j=1}^{\infty}$ be a Cauchy sequence in ℓ^1.

Write $x^{(j)} = (x_n^{(j)})_{n=-\infty}^{\infty}$ for every $j \geq 1$.

Convention: $x^{(1)}, x^{(2)}, \ldots$ is a Cauchy sequence in ℓ^1, so $x^{(j)}$ is the jth element in your Cauchy seq. And $x_n^{(j)}$ is the nth term of $x^{(j)}$.

Want to construct $x \in \ell^1$ such that
$\| x^{(j)} - x \|_{\ell^1} \longrightarrow 0$ as $j \to +\infty$.

By passing to a subsequence, we may assume that $\| x^{(j)} - x^{(k)} \|_{\ell^1} \leq \frac{1}{2^{\min\{j,k\}}}$, $\forall j,k \geq 1$.
(see Homework 1).

i.e. $\sum_{n=-\infty}^{\infty} | x_n^{(j)} - x_n^{(k)} | \leq \frac{1}{2^{\min\{j,k\}}}$ $\forall j,k \geq 1$.

Then $\forall n \in \mathbb{Z}$, the sequence $(x_n^{(j)})_{j=1}^{\infty}$ is Cauchy in $\mathbb{C}$. (because $\forall \varepsilon > 0$, choose $M \in \mathbb{N}$ such that $\frac{1}{2^M} < \varepsilon$. Then $\forall j,k \geq M$, $| x_n^{(j)} - x_n^{(k)} | \leq \| x^{(j)} - x^{(k)} \|_{\ell^1} \leq 2^{-M} < \varepsilon$.)

$$x^{(1)} = \left(\cdots, x^{(1)}_{-2}, x^{(1)}_{-1}, x^{(1)}_{0}, x^{(1)}_{1}, x^{(1)}_{2}, \cdots \right)$$

$$x^{(2)} = \left(\cdots, x^{(2)}_{-2}, x^{(2)}_{-1}, x^{(2)}_{0}, x^{(2)}_{1}, x^{(2)}_{2}, \cdots \right)$$

$$\vdots$$

$$\downarrow$$

$$x_1$$

So for each $n \in \mathbb{Z}$, we can let $x_n := \lim_{j \to +\infty} x^{(j)}_n$.

(Possible because $(x^{(j)}_n)_{j=1}^{\infty}$ is Cauchy in $\mathbb{C}$).

And now set $x = (x_n)_{n=-\infty}^{\infty}$.

Need to check: (a) $x \in \ell^1(\mathbb{Z})$

(b) $\| x^{(j)} - x \|_{\ell^1} \longrightarrow 0$ as $j \to +\infty$.

Check (b).

$$\|x^{(j)} - x\|_{\ell^1} = \lim_{N\to\infty} \sum_{n=-N}^{N} |x_n^{(j)} - x_n|$$

We want this to be small as $j \to +\infty$.

But

$$\sum_{n=-N}^{N} |x_n^{(j)} - x_n| = \lim_{k\to\infty} \sum_{n=-N}^{N} |x_n^{(j)} - x_n^{(k)}|$$

$$\leq \lim_{k\to\infty} \|x^{(j)} - x^{(k)}\|_{\ell^1}$$

$$\leq \frac{1}{2^j} \qquad \forall j, N \geq 1$$

So let $N \to \infty$, we get $\|x^{(j)} - x\|_{\ell^1} \leq \frac{1}{2^j} \quad \forall j \geq 1$

So $(x^{(j)})$ converges to x in $\ell^1(\mathbb{Z})$ as $j \to \infty$.

∎

④ $\ell^2(\mathbb{Z}) = \{ (x_n)_{n=-\infty}^{\infty} : \sum_{n=-\infty}^{\infty} |x_n|^2 < \infty \}$

with $\|(x_n)\|_{\ell^2} := \left(\sum_{n=-\infty}^{\infty} |x_n|^2 \right)^{1/2}$

Then $\ell^2(\mathbb{Z})$ is complete. (Exercise).

An incomplete metric space can always be completed abstractly, but in practice it is often more useful to have a concrete construction of the completion. For $(C^0[0,1], L^1)$, $(C^0[0,1], L^2)$, the completion can be constructed using Lebesgue integration. For $(C^0[0,1], L^2)$, an alternative way of completion is via its embedding into $\ell^2(\mathbb{Z})$, using Fourier series. We will construct an injective, norm-preserving linear map $(C^0[0,1], L^2) \longrightarrow \ell^2(\mathbb{Z})$ in the next two lectures.