Analysis I Add on Lecture 6

Fourier series : Part 1

Last time: $(C^0[0,1], \|\cdot\|_{L^2})$ is not complete
whereas $\ell^2(\mathbb{Z})$ is complete under $\|\cdot\|_{\ell^2}$.
But $(C^0, \|\cdot\|_{L^2})$ can be embedded into $\ell^2(\mathbb{Z})$ isometrically, so $\ell^2(\mathbb{Z})$ can be taken as the completion of $(C^0, \|\cdot\|_{L^2})$. For that embedding we describe it via Fourier series

Recall $\|f\|_{L^2} = \left(\int_0^1 |f(x)|^2 dx\right)^{1/2}$ for $f \in C^0([0,1])$

This norm actually comes from an inner product on $C^0[0,1]$

namely $\langle f, g\rangle := \int_0^1 f(x)\overline{g(x)}\, dx$ (recall: $\int_0^1 (u(x)+iv(x))dx = \int_0^1 u(x) + i\int_0^1 v(x)dx$)

Example. $e^{ix} := \cos x + i \sin x$ for $x \in \mathbb{R}$.

$$\int_0^{2\pi} e^{ix}\,dx = \int_0^{2\pi} \cos x\,dx + i\int_0^{2\pi} \sin x\,dx$$

$$= \sin x \Big|_0^{2\pi} + i\,(-\cos x)\Big|_0^{2\pi}$$

$$= 0 + i\,0 = 0.$$

Exercise. $\int_0^1 e^{2\pi i n x}\,dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \in \mathbb{Z}\setminus\{0\}. \end{cases}$

$$\left(\int_0^1 e^{2\pi i n x}\,dx = \int_0^1 \cos(2\pi n x)\,dx + i\int_0^1 \sin(2\pi n x)\,dx \right)$$

Also recall that $\bar{z} = x - iy$ if $z = x + iy$

$$|z| = (z\bar{z})^{1/2}$$

So now we can make sense of $\langle f, g\rangle = \int_0^1 f(x)\,\overline{g(x)}\,dx$

In particular, $\langle f, f \rangle = \int_0^1 |f(x)|^2 \, dx = \|f\|_{L^2}^2$

Starting point : $\{e^{2\pi i n x}\}_{n \in \mathbb{Z}}$ is an orthonormal set in $C^0[0,1]$ under this inner product.

i.e. $\langle e^{2\pi i n x}, e^{2\pi i m x} \rangle = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$, $m, n \in \mathbb{Z}$

This follows since

$$\langle e^{2\pi i n x}, e^{2\pi i m x} \rangle = \int_0^1 e^{2\pi i n x} \, \overline{e^{2\pi i m x}} \, dx$$

$$= \int_0^1 e^{2\pi i n x} e^{-2\pi i m x} \, dx$$

$$= \int_0^1 e^{2\pi i (n-m) x} \, dx$$

$$= \begin{cases} 1 & \text{if } n - m = 0 \\ 0 & \text{if } n - m \neq 0 \end{cases} \quad \text{by Exercise.}$$

Question. Is $\{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ an orthonormal "basis"?

Can every $f \in C^0[0,1]$ be expanded as

$$f = \sum_{n\in\mathbb{Z}} \langle f, e^{2\pi i n x}\rangle e^{2\pi i n x} \ ?$$

What does this mean? How do you interpret / understand the infinite sum on the right hand side?

Definition. For $f \in C^0[0,1]$ and $n \in \mathbb{Z}$,

let $\hat{f}(n) = \langle f, e^{2\pi i n x}\rangle = \int_0^1 f(x) e^{-2\pi i n x}\, dx$

For $N \in \mathbb{N}$, let $P_N = \operatorname{span}\{e^{2\pi i n x}\}_{n=-N}^{N}$

$= \left\{ \sum_{n=-N}^{N} a_n e^{2\pi i n x} \;\middle|\; a_{-N}, a_{-N+1}, \dots, a_N \in \mathbb{C} \right\}$

$=$ space of trigonometric polynomials of degree $\leq N$

Let $S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x} \in P_N$ "Nth partial sum of Fourier series of f".

Lemma. Let $f \in C^0[0,1]$, $N \in \mathbb{N}$. Then $S_N f$ is the orthogonal projection of f onto P_N, so that

$$\|f - S_N f\|_{L^2} = \inf_{P \in P_N} \|f - P\|_{L^2}$$

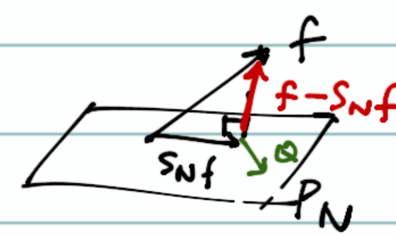
(i.e. $\|f - S_N f\|_{L^2} \le \|f - P\|_{L^2} \quad \forall\, P \in P_N$.)

Proof. $S_N f = \sum_{n=-N}^{N} \langle f, e^{2\pi i n x}\rangle e^{2\pi i n x}$ and $\{e^{2\pi i n x}\}_{n=-N}^{N}$ is an orthonormal basis of P_N.

So $S_N f$ is the orthogonal projection of f onto P_N

If $Q(x) = \sum_{m=-N}^{N} c_m e^{2\pi i m x}$ is any element of P_N, then we claim

$$\langle f - S_N f, Q \rangle = 0.$$



In fact, $\langle f - S_N f, Q \rangle$

$$= \langle f - \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x}, \sum_{m=-N}^{N} c_m e^{2\pi i m x} \rangle$$

$$= \langle f, \sum_{m=-N}^{N} c_m e^{2\pi i m x} \rangle - \sum_{n=-N}^{N} \hat{f}(n) \langle e^{2\pi i n x}, \sum_{m=-N}^{N} c_m e^{2\pi i m x} \rangle$$

$$= \sum_{m=-N}^{N} \overline{c_m} \langle f, e^{2\pi i m x} \rangle - \sum_{n=-N}^{N} \hat{f}(n) \overline{c_n}$$

$$= \sum_{m=-N}^{N} \overline{c_m} \hat{f}(m) - \sum_{n=-N}^{N} \hat{f}(n) \overline{c_n}$$

$$= 0.$$

In particular, if $P \in P_N$, then

$$\langle f - S_N f, S_N f - P \rangle = 0$$

As a result

$$
\begin{aligned}
\| f - P \|_{L^2}^2 &= \| (f - S_N f) + (S_N f - P) \|_{L^2}^2 \\
&= \| f - S_N f \|_{L^2}^2 + \| S_N f - P \|_{L^2}^2 \\
&\quad + 2 \operatorname{Re} \underbrace{\langle f - S_N f, S_N f - P \rangle}_{0} \\
&= \| f - S_N f \|_{L^2}^2 + \| S_N f - P \|_{L^2}^2 \\
&\geq \| f - S_N f \|_{L^2}^2
\end{aligned}
$$

This is true for every $P \in P_N$. Hence

$$
\| f - S_N f \|_{L^2} = \inf_{P \in P_N} \| f - P \|_{L^2} .
$$

$\blacksquare$

Next, we'll show that for $f \in C^0[0,1]$,

$$\inf_{P \in P_N} \|f - P\|_{L^2} \longrightarrow 0 \quad \text{as } N \to \infty. \qquad \text{—— } (*)$$

If we prove this, then by lemma, we see that

$$\|f - S_N f\|_{L^2} \longrightarrow 0 \quad \text{as } N \to \infty$$

i.e. $\lim_{N \to \infty} S_N f = f$ in L^2.

$$\lim_{N\to\infty} S_N f = \lim_{N \to \infty} \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x}$$

Indeed, let $\sigma_N f = \frac{1}{N+1} \sum_{n=0}^{N} S_n f \in P_N$

If we can show that $S_N f \to f$ in L^2 as $N \to \infty$ then $\sigma_N f$ also tend to f in L^2 as $N \to \infty$. But the latter is easier to show.

Theorem. For $f \in C^0[0,1]$, if $f(0) = f(1)$, then

$$\| f - \sigma_N f \|_{C^0} \longrightarrow 0 \quad \text{as } N \to \infty$$

Using this, and an easy approximation argument, next time we'll prove that $\inf_{p \in P_N} \| f - p \|_{L^2} \longrightarrow 0$ as $N \to \infty$ $\forall f \in C^0[0,1]$,

and hence $\| S_N f - f \|_{L^2} \longrightarrow 0$ as $N \to \infty$ $\forall f \in C^0[0,1]$.