Analysis 1 Add on Lecture 7

Fourier Series (Part 2).

Announcement:
Typos in Homework 1 corrected. Please download latest version from Wattle. (particularly in Q1(d))

Last time: Fourier series of $f \in C^0[0,1]$

For $f \in C^0[0,1]$, define $\hat{f}(n) := \int_0^1 f(x)\, e^{-2\pi i n x}\, dx$ for $n \in \mathbb{Z}$.

We defined $S_N f(x) := \sum_{|n| \le N} \hat{f}(n) e^{2\pi i n x}$ "Nth partial sum of Fourier series of f".

Question: Does $\| S_N f - f \|_{L^2([0,1])} \to 0$ as $N \to \infty$?

From last time: $\| S_N f - f \|_{L^2} = \inf_{P \in P_N} \| f - P \|_{L^2}$

where $P_N = \mathrm{span}\{ e^{2\pi i n x} : n = -N, \dots, N \}$. the space of all trigonometric polynomials of degree $\le N$.

Today: $\inf_{P \in P_N} \| f - P \|_{L^2} \to 0$ as $N \to \infty$.

Define $\sigma_N f(x) := \frac{1}{N+1} \sum_{n=0}^{N} S_n f(x)$ "average of $S_0 f, S_1 f, \ldots, S_N f$"

$\in P_N$

Theorem If $g \in C^0[0,1]$ and $g(0) = g(1)$, then

$$\| \sigma_N g - g \|_{C^0} \to 0 \quad \text{as } N \to \infty$$

(recall $\|\sigma_N g - g\|_{C^0} = \sup_{x \in [0,1]} |\sigma_N g(x) - g(x)|$.)

As a result, the answer to the previous question is yes: Given $f \in C^0[0,1]$, and given $\varepsilon > 0$, choose $g \in C^0[0,1]$ with $g(0) = g(1)$ so that $\| f - g \|_{L^2[0,1]} < \varepsilon$.

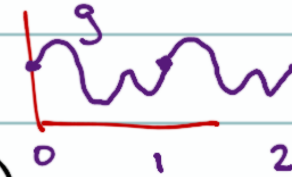
Then $\|\sigma_N f - f\|_{L^2} \le \|\sigma_N (f-g)\|_{L^2} + \|f - g\|_{L^2} + \|\sigma_N g - g\|_{L^2}$

$$\le 2\varepsilon + \|\sigma_N g - g\|_{C^0}$$

$$\le 3\varepsilon \quad \text{if } N \text{ is large, using Theorem.}$$

Hence $\|\sigma_N f - f\|_{L^2[0,1]} \to 0$ as $N \to \infty$.

Proof of Theorem. Let $g \in C^0[0,1]$ with $g(0) = g(1)$.



Extend g periodically to $\mathbb{R}$ so that $g \in C^0(\mathbb{R})$

$g(x+1) = g(x)$ $\forall\, x \in [0,1]$, etc...

Key: $$\sigma_N g(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(x-y)\, F_N(y)\, dy$$

where $$F_N(y) := \frac{1}{N+1} \sum_{n=0}^{N} \sum_{m=-n}^{n} e^{2\pi i m y} = \frac{1}{N+1} \frac{\sin^2(\pi(N+1)y)}{\sin^2(\pi y)}.$$

"Fejer kernel".

Indeed, $$\sigma_N g(x) = \frac{1}{N+1} \sum_{n=0}^{N} S_n g(x)$$

$$= \frac{1}{N+1} \sum_{n=0}^{N} \sum_{m=-n}^{n} \hat{g}(m) e^{2\pi i m x}$$

$$= \frac{1}{N+1} \sum_{n=0}^{N} \sum_{m=-n}^{n} \int_0^1 g(y) e^{-2\pi i m y}\, dy \cdot e^{2\pi i m x}$$

$$= \int_0^1 g(y) \left[\frac{1}{N+1} \sum_{n=0}^{N} \sum_{m=-n}^{n} e^{2\pi i m (x-y)} \right] dy$$

Apply change of variables $y \mapsto x-y$, use periodicity.

Indeed, $\sigma_N f(x) = \int_0^1 g(y)\, F_N(x-y)\, dy$

$= \int_{x-1}^{x} g(x-y)\, F_N(y)\, dy$ (change of variable $y \mapsto x-y$)

$= \int_0^1 g(x-y)\, F_N(y)\, dy$ (periodicity of g and F_N).

where $F_N(y) = \frac{1}{N+1} \sum_{n=0}^{N} \sum_{m=-n}^{n} e^{2\pi i m y}$.

But one can check that $F_N(y) = \frac{1}{N+1} \frac{\sin^2(\pi(N+1)y)}{\sin^2(\pi y)}$:
this is because $F_N(y)$ is sum of geometric series.
Recall $e^{ix} := \cos x + i \sin x$ if $x \in \mathbb{R}$. Using compound angle formula, one can check that $e^{ix} \cdot e^{iy} = e^{i(x+y)}$ $\forall x, y \in \mathbb{R}$.
Hence $e^{2\pi i m y} = \left(e^{2\pi i y}\right)^m$ $\forall m \in \mathbb{Z}, y \in \mathbb{R}$.

Hence $F_N(y) = \frac{1}{N+1} \sum_{n=0}^{N} \sum_{m=-n}^{n} e^{2\pi i m y}$

$$= \frac{1}{N+1} \sum_{n=0}^{N} \frac{e^{2\pi i(n+1)y} - e^{-2\pi i n y}}{e^{2\pi i y} - 1}$$

$$= \frac{1}{N+1} \left[\frac{e^{2\pi i y}\left[e^{2\pi i (N+1)y} - 1\right]}{\left(e^{2\pi i y} - 1\right)^2} - \frac{1 - e^{-2\pi i (N+1)y}}{\left(1 - e^{-2\pi i y}\right)\left(e^{2\pi i y} - 1\right)} \right]$$

$$= \frac{1}{N+1} \frac{e^{2\pi i (N+1)y} - 2 + e^{-2\pi i (N+1)y}}{\left(e^{\pi i y} - e^{-\pi i y}\right)^2}$$

$$= \frac{1}{N+1} \frac{\left(e^{\pi i (N+1)y} - e^{-\pi i (N+1)y}\right)^2}{\left(e^{\pi i y} - e^{-\pi i y}\right)^2}$$

$$= \frac{1}{N+1} \frac{\sin^2(\pi(N+1)y)}{\sin^2(\pi y)}.$$

Recap: If $g \in C^0[0,1]$ with $g(0)=g(1)$, then

$$\sigma_N g(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(x-y)\, F_N(y)\, dy$$

where $F_N(y) = \frac{1}{N+1} \sum_{n=0}^{N} \sum_{m=-n}^{n} e^{2\pi i m y} = \frac{1}{N+1} \frac{\sin^2(\pi(N+1)y)}{\sin^2(\pi y)}$.

Properties. ① $\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(y)\, dy = 1$

② $F_N(y) \geq 0 \quad \forall y$

③ For any $\delta \in (0, \frac{1}{2})$, $\int_{\delta < |y| < \frac{1}{2}} |F_N(y)|\, dy \to 0$ as $N \to \infty$.

$\int_{-\frac{1}{2}}^{-\delta} + \int_{\delta}^{\frac{1}{2}}$

Proof. ① $\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(y)\, dy = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{N+1} \sum_{n=0}^{N} \sum_{m=-n}^{n} e^{2\pi i m y}\, dy = \frac{1}{N+1} \sum_{n=0}^{N} 1 = 1.$

(For $m \in \mathbb{Z}$, $\int_0^1 e^{2\pi i m y}\, dy = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases}$)

② $F_N(y) = \frac{1}{N+1} \frac{\sin^2(\cdots)}{\sin^2(\cdots)} \geq 0$

③. For $\delta \in (0, \frac{1}{2})$, $\int_{\delta < |y| < \frac{1}{2}} |F_N(y)|\, dy \leq \frac{1}{N+1} \int_{\delta < |y| < \frac{1}{2}} \frac{1}{\sin^2(\pi\delta)}\, dy \to 0$ as $N \to \infty$.

Want to prove

$$\|\sigma_N g - g\|_{C^0} = \sup_{x\in[0,1]} |\sigma_N g(x) - g(x)| \to 0 \quad \text{as} \quad N \to \infty.$$

since $\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(y) = 1.$

But for $x \in [0,1]$,

$$\sigma_N g(x) - g(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(x-y) F_N(y)\, dy - \int_{-\frac{1}{2}}^{\frac{1}{2}} g(x) F_N(y) dy$$

$$\text{So } |\sigma_N g(x) - g(x)| \le \int_{-\frac{1}{2}}^{\frac{1}{2}} |g(x-y) - g(x)|\, F_N(y)\, dy$$

no absolute value because $F_N(y) \geq 0$.

$$= \int_{-\delta}^{\delta} \underline{|g(x-y) - g(x)|}\, F_N(y)\, dy + \int_{\delta<|y|<\frac{1}{2}} \underline{|g(x-y) - g(x)|}\, F_N(y) dy$$

Given $\varepsilon > 0$, choose $\delta > 0$ so that $|g(x-y) - g(x)| \le \varepsilon$ $\forall\, x \in [0,1]$ and $|y| < \delta$ (Possible by uniform continuity of g on $[0,1]$). Also let $M = \sup_{x\in[0,1]} |g(x)|$.

Then from above, $|\sigma_N g(x) - g(x)| \le \int_{-\delta}^{\delta} \varepsilon\, F_N(y)\, dy + \int_{\delta<|y|<\frac{1}{2}} 2M\, F_N(y)\, dy$

$$< \varepsilon \cdot 1 + 2M \cdot \frac{\varepsilon}{2M} \text{ if } N \text{ is large enough, by Property ③.}$$

We have proved that

$$\sup_{x\in[0,1]} |\sigma_N g(x) - g(x)| < 2\varepsilon \quad \text{if } N \text{ is large enough,}$$

thus $\quad \|\sigma_N g - g\|_{C^0[0,1]} \to 0 \quad$ as $\quad N \to \infty$.

This finishes the proof of the theorem today, and

hence $\quad \| S_N f - f \|_{L^2[0,1]} \to 0 \quad$ as $\quad N \to \infty, \quad \forall f \in C^0[0,1]$.

<u>Remarks</u>: ① For $f \in C^0[0,1]$, $S_N f$ may not converge uniformly to f on $[0,1]$, even if $f(0) = f(1)$.

② From the above, we have, for every $f \in C^0[0,1]$, that

$$\|f\|_{L^2} = \lim_{N\to\infty} \| S_N f \|_{L^2} = \lim_{N\to\infty} \left(\sum_{n=-N}^{N} |\hat{f}(n)|^2 \right)^{1/2}.$$

(Parseval's identity; e.g. if $f(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}] \\ 1-x & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$

then Parseval $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.)

Hence $(C^0[0,1], L^2) \hookrightarrow \ell^2(\mathbb{Z})$ is an isometric embedding,

$$f \longmapsto \{\hat{f}(n)\}_{n\in\mathbb{Z}}$$

and since $\ell^2(\mathbb{Z})$ is complete, we can think of $\ell^2(\mathbb{Z})$ as the completion of $(C^0[0,1], L^2)$.

Next time: More about the inner product on $L^2[0,1]$; Cauchy-Schwarz, etc.