Analysis I Add on Lecture 8.

Inequalities (Part 1) : Minkowski and Hölder's inequality.

We worked with L^2 norm on $C^0[0,1]$ in the previous lectures, and sometimes we use the L^p norm

$$\|f\|_{L^p} := \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}} \quad \text{for } f \in C^0[0,1], \; 1 \le p < \infty.$$

<u>Question</u> Why is this a norm?

In particular, why does it satisfy the triangle inequality?

i.e. Why is true that $\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$?

Today: First show that

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}, \text{ if } f, g \text{ are}$$

Riemann integrable on $[0,1]$,

and $1 \leq p < \infty$.

i.e. $$\left(\int_0^1 |f(x)+g(x)|^p \, dx\right)^{\frac{1}{p}} \leq \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_0^1 |g(x)|^p dx\right)^{\frac{1}{p}}.$$

"Minkowski's inequality"

Quoting Pólya: What is the simplest problem that you cannot solve?

So let's try something less ambitions:

Can you prove that $\int_0^1 |f(x)+g(x)|^p \, dx < \infty$, if both $\int_0^1 |f(x)|^p dx$ and $\int_0^1 |g(x)|^p dx < \infty$?

Sure!

Consider first the case $p=1$.

$$|f(x)+g(x)| \leq |f(x)| + |g(x)| \qquad \forall x \in [0,1]$$

$$\Longrightarrow \int_0^1 |f(x)+g(x)|\,dx \leq \int_0^1 |f(x)|\,dx + \int_0^1 |g(x)|\,dx.$$

This proves the Minkowski inequality for $p=1$.

What about $p>1$? Reduce to something you know. Mimic case $p=1$.

Pointwisely,

$$\begin{aligned} |f(x)+g(x)|^p &\leq \left(|f(x)|+|g(x)|\right)^p \\ &\leq \left(2\max\{|f(x)|,|g(x)|\}\right)^p \\ &\leq 2^p\left(|f(x)|^p + |g(x)|^p\right) \end{aligned}$$

$$\Longrightarrow \int_0^1 |f(x)+g(x)|^p dx \leq 2^p\left(\int_0^1 |f(x)|^p dx + \int_0^1 |g(x)|^p dx\right)$$

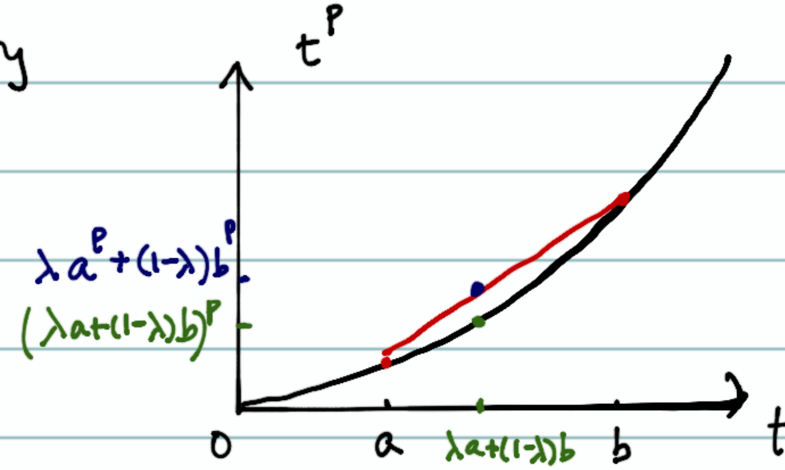
$$\Longrightarrow \left(\int_0^1 |f(x)+g(x)|^p dx\right)^{\frac{1}{p}} \leq 2\left(\int_0^1 |f(x)|^p dx + \int_0^1 |g(x)|^p dx\right)^{\frac{1}{p}}$$

$(A+B)^{\frac{1}{p}} \leq A^{\frac{1}{p}} + B^{\frac{1}{p}}$ $\forall A,B \geq 0,\ p>1.$

$$\leq 2\left[\left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_0^1 |g(x)|^p dx\right)^{\frac{1}{p}}\right]$$

Problem: extra 2 on RHS!

Refine this estimate! Idea: convexity



$t \mapsto t^p$ is convex on $[0,\infty)$

if $1<p<\infty$

$1<p<\infty$

$0<\lambda<1$

$\Rightarrow$ $\forall \lambda \in (0,1)$, $\forall a,b \geq 0$,

we have

$$(\lambda a+(1-\lambda)b)^p \leq \lambda a^p + (1-\lambda)b^p .$$

Replacing a by $\frac{a}{\lambda}$ and b by $\frac{b}{1-\lambda}$, we get

$$(a+b)^p \leq \lambda^{1-p} a^p + (1-\lambda)^{1-p} b^p .$$

Hence $\begin{cases} \forall x \in [0,1] \\ \forall \lambda \in (0,1) \end{cases}$, $|f(x)+g(x)|^p \leq (|f(x)|+|g(x)|)^p$

$$\leq \lambda^{1-p}|f(x)|^p + (1-\lambda)^{1-p}|g(x)|^p$$

$\forall \lambda \in (0,1)$, $\forall x \in [0,1]$, $|f(x)+g(x)|^p \le \lambda^{1-p}|f(x)|^p + (1-\lambda)^{1-p}|g(x)|^p$.

$$\Rightarrow \forall \lambda \in (0,1), \quad \int_0^1 |f(x)+g(x)|^p\,dx \le \lambda^{1-p}\underbrace{\int_0^1 |f(x)|^p\,dx}_{\|f\|_{L^p}^p} + (1-\lambda)^{1-p}\underbrace{\int_0^1 |g(x)|^p\,dx}_{\|g\|_{L^p}^p}$$

Extra flexibility in choosing $\lambda \in (0,1)$:

Choose the best $\lambda \in (0,1)$! Choose $\lambda \in (0,1)$ to minimize the RHS.

If $\|f\|_{L^p}$ and $\|g\|_{L^q}$ are both non-zero, then this happens precisely when $\lambda = \dfrac{\|f\|_{L^p}}{\|f\|_{L^p}+\|g\|_{L^p}}$,

in which case $\lambda^{1-p}\|f\|_{L^p}^p + (1-\lambda)^{1-p}\|g\|_{L^p}^p$

$$= \frac{\|f\|_{L^p}^{1-p+p}}{(\|f\|_{L^p}+\|g\|_{L^p})^{1-p}} + \frac{\|g\|_{L^p}^{1-p+p}}{(\|f\|_{L^p}+\|g\|_{L^p})^{1-p}}$$

$$= (\|f\|_{L^p}+\|g\|_{L^p})^{p-1}\left[\|f\|_{L^p}+\|g\|_{L^p}\right] = (\|f\|_{L^p}+\|g\|_{L^p})^p.$$

$$\Rightarrow \|f+g\|_{L^p}^p \le (\|f\|_{L^p}+\|g\|_{L^p})^p$$

$$\Rightarrow \|f+g\|_{L^p} \le \|f\|_{L^p}+\|g\|_{L^p}.$$

(If $\|f\|_{L^p}$ or $\|g\|_{L^p}=0$ then $\|f+g\|_{L^p} \le \|f\|_{L^p}+\|g\|_{L^p}$ trivially.)

C.f. Workshop problem:

Prove Minkowski's inequality using Hölder's inequality, which says that

$$\int_0^1 |f(x)\, g(x)|\, dx \leq \|f\|_{L^p} \|g\|_{L^q}$$

if f, g are Riemann integrable on $[0,1]$, and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

But we can reverse the logic, and prove Hölder's inequality using Minkowski's inequality!

Key insight:

If $\begin{cases} 1 < p, q < \infty \\ \frac{1}{p} + \frac{1}{q} = 1 \end{cases}$, and if $A, B \geq 0$, then

$$\lim_{s \to \infty} \left(\frac{A^{p/s}}{p} + \frac{B^{q/s}}{q} \right)^s = AB.$$

"$\left(\frac{A^0}{p} + \frac{B^0}{q}\right)^\infty = 1^\infty$"

→ use L'Hopital's rule.

(proof omitted).

To prove Hölder, without loss of generality assume $f, g \geq 0$ on $[0,1]$.

Now from Minkowski's inequality for L^s,

$$\left\| \frac{f^{p/s}}{p} + \frac{g^{q/s}}{q} \right\|_{L^s} \leq \frac{1}{p} \|f^{p/s}\|_{L^s} + \frac{1}{q} \|g^{q/s}\|_{L^s}$$

$$= \frac{1}{p} \|f\|_{L^p}^{p/s} + \frac{1}{q} \|g\|_{L^q}^{q/s}$$

$$\Rightarrow \int_0^1 \left(\frac{f(x)^{p/s}}{p} + \frac{g(x)^{q/s}}{q} \right)^s dx \leq \left(\frac{1}{p} \|f\|_{L^p}^{p/s} + \frac{1}{q} \|g\|_{L^q}^{q/s} \right)^s$$

Let $s \to \infty$, $\int_0^1 f(x) g(x)\, dx \leq \|f\|_{L^p} \|g\|_{L^q}$. Hölder's!

Corollary: (Cauchy-Schwarz)

$$\int_0^1 |f(x)g(x)|\,dx \le \|f\|_{L^2} \|g\|_{L^2}$$

whenever f, g are Riemann integrable on $[0,1]$

(just take $p = q = 2$ in Hölder's).