Analysis 1 Add on Lecture 9

Inequalities (Part 2): Hölder, Convexity and Young's inequality.

Recall from last time:

We proved Hölder's inequality, i.e.

$$\int_0^1 f(x)g(x)\,dx \le \|f\|_{L^p} \|g\|_{L^q} \quad \text{if } 1 < p, q < \infty, \ \frac{1}{p} + \frac{1}{q} = 1$$

Using Minkowski's inequality:

$$\|f+g\|_{L^s} \le \|f\|_{L^s} + \|g\|_{L^s}, \quad \text{if } 1 \le s < \infty$$

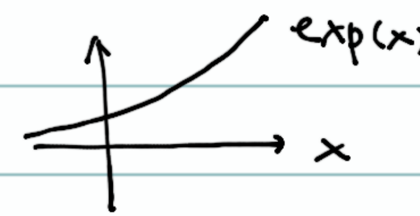
Here $\|f\|_{L^p} = \left(\int_0^1 |f(x)|^p\,dx\right)^{\frac{1}{p}}$ for $1 \le p < \infty$

This time: try to better understand proof from last time

Idea: To go from Minkowski to Hölder, we need to pass from a sum (of functions) to product (of functions).

. The exponential function $\exp(x+y) = \exp(x)\exp(y)$ should be called for!

Idea: One knows Hölder implies Minkowski, and the proof of Minkowski last time involves convexity. Also exp is convex! Exploit this convexity!



First, some generalities about C^2 convex functions.

(A C^2 function f is said to be convex on an interval I, if $f'' \geq 0$ on I).

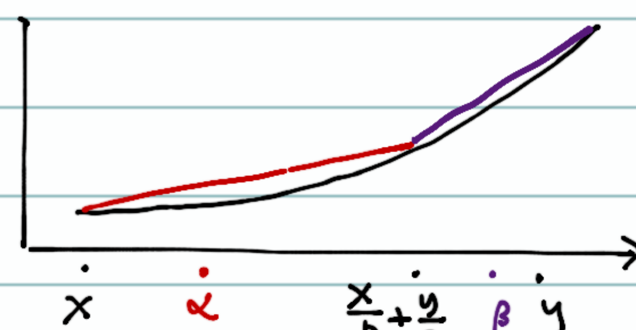
If f is C^2, $\begin{cases} f'' \geq 0 & \text{on } [x,y] \\ f'' \leq M & \text{on } [x,y] \end{cases}$, and if $\begin{cases} 1 < p, q < \infty \\ \frac{1}{p} + \frac{1}{q} = 1 \end{cases}$

then we claim $$f\left(\frac{x}{p}+\frac{y}{q}\right) \leq \frac{f(x)}{p} + \frac{f(y)}{q} \leq f\left(\frac{x}{p}+\frac{y}{q}\right) + \frac{M(y-x)^2}{pq}.$$

This is because:

① $f\left(\frac{x}{p}+\frac{y}{q}\right) = f(x) + f'(\alpha)\cdot\frac{y-x}{q}$ for some α between x and $\frac{1}{p}x+\frac{1}{q}y$

② $f\left(\frac{x}{p}+\frac{y}{q}\right) = f(y) - f'(\beta)\cdot\frac{y-x}{p}$ for some β between y and $\frac{1}{p}x+\frac{1}{q}y$



$\frac{x}{p}+\frac{y}{q}-x = \frac{y-x}{q}$ $\qquad$ $y - \left(\frac{x}{p}+\frac{y}{q}\right) = \frac{y-x}{p}$.

$\frac{①}{p} + \frac{②}{q}$: $$f\left(\frac{x}{p}+\frac{y}{q}\right) = \frac{f(x)}{p} + \frac{f(y)}{q} - \left(f'(\beta)-f'(\alpha)\right)\left(\frac{y-x}{pq}\right)$$

In particular, $$\frac{f(x)}{p} + \frac{f(y)}{q} - f\left(\frac{x}{p}+\frac{y}{q}\right)$$

$$= \left(f'(\beta)-f'(\alpha)\right)\left(\frac{y-x}{pq}\right) \begin{cases} \geqslant 0 & \text{since } f'' \geqslant 0 \text{ on } [x,y] \\ \leq \dfrac{M(y-x)^2}{pq} & \text{since } f'' \leq M \text{ on } [x,y] \end{cases}$$

(indeed, $f'(\beta)-f'(\alpha) \leq M(\beta-\alpha) \leq M(y-x)$.)

giving

$$f\left(\frac{x}{p}+\frac{y}{q}\right) \leq \frac{f(x)}{p} + \frac{f(y)}{q} \leq f\left(\frac{x}{p}+\frac{y}{q}\right) + \frac{M(y-x)^2}{pq}$$

as claimed.

Applying this to $f(x) = \exp(x)$ on $[-1,1]$:
(with $M = e$)
it follows that

$$\exp\left(\frac{x}{p}+\frac{y}{q}\right) \leq \frac{\exp(x)}{p} + \frac{\exp(y)}{q} \leq \exp\left(\frac{x}{p}+\frac{y}{q}\right) + e\,\frac{(y-x)^2}{pq}$$

$$\forall x, y \in [-1,1]$$

$$(*): \quad \exp\left(\frac{x}{p}+\frac{y}{q}\right) \le \frac{\exp(x)}{p} + \frac{\exp(y)}{q} \le \exp\left(\frac{x}{p}+\frac{y}{q}\right) + e\,\frac{(y-x)^2}{pq}$$

$$\forall x,y \in [-1,1].$$

We use this to revisit the key inequality from last time:

If $\begin{cases} 1<p,q<\infty \\ \frac{1}{p}+\frac{1}{q}=1 \end{cases}$ and $A,B \geq 0$, then

Going from a sum on LHS to the product on RHS.

$$\lim_{s\to+\infty}\left(\frac{A^{p/s}}{p}+\frac{B^{q/s}}{q}\right)^s = AB.$$

WLOG assume $A,B>0$ (otherwise this limit is trivial)

$\frac{x}{p}+\frac{y}{q} = \log(AB)^{\frac{1}{s}}$

Apply $(*)$ to $x = \frac{\log A^p}{s}$ and $y = \frac{\log B^q}{s}$ for s large

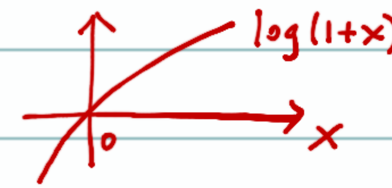
$y-x = \frac{\log B^q - \log A^p}{s}$

then $(AB)^{\frac{1}{s}} \le \frac{A^{p/s}}{p} + \frac{B^{q/s}}{q} \le (AB)^{\frac{1}{s}} + \frac{C}{s^2}$

for some constant $C \geq 0$ that depends only on p,q,A,B.

Hence
$$
\begin{aligned}
AB &\le \left(\frac{A^{p/s}}{p} + \frac{B^{q/s}}{q}\right)^s \le \left((AB)^{\frac{1}{s}} + \frac{C}{s^2}\right)^s \\
&= AB\left(1 + \frac{C}{(AB)^{\frac{1}{s}} s^2}\right)^s \\
&= AB \exp\left(s \log\left(1 + \frac{C}{(AB)^{\frac{1}{s}} s^2}\right)\right) \\
&\le AB \exp\left(\cancel{s} \frac{C}{(AB)^{\frac{1}{s}} s^{\cancel{2}}}\right) \\
&\to AB \exp(0) = AB \quad \text{as } s \to +\infty.
\end{aligned}
$$

since $\log(1+x) \le x \quad \forall x \ge 0$



From sandwich theorem, $\lim_{s \to +\infty} \left(\frac{A^{p/s}}{p} + \frac{B^{q/s}}{q}\right)^s = AB$,
giving another proof of the key inequality from last time.