Math 252 Spring 2012
Notes about differential operators and second order equations

A differential operator is an operator that takes a function to its derivatives. For example, the operator $\frac{d}{dt}$ sends a function $y(t)$ to its derivative $\frac{dy}{dt}$. More generally, if a, b are two constants, we can consider an operator

$$a\frac{d}{dt} + b;$$

this operator sends a function $y(t)$ to $a\frac{dy}{dt} + by(t)$. Symbolically, we write

$$\left(a\frac{d}{dt} + b\right) y(t) = a\frac{dy}{dt} + by(t).$$

An operator of the form $a\frac{d}{dt} + b$ is called a first order differential operator with constant coefficients. We can also consider second order differential operators with constant coefficients; they are of the form

$$a\frac{d^2}{dt^2} + b\frac{d}{dt} + c$$

where a, b, c are constants, and they send a function $y(t)$ to

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy(t).$$

Two differential operators can be composed in the obvious way: for example, if r_1, r_2 are two constants, we can compose the differential operators $\frac{d}{dt} - r_1$ with $\frac{d}{dt} - r_2$ by letting them act on a function $y(t)$ successively. In other words,

$$\begin{aligned}\left(\frac{d}{dt} - r_1\right)\left(\frac{d}{dt} - r_2\right) y(t) &= \left(\frac{d}{dt} - r_1\right)\left(\frac{dy}{dt} - r_2 y(t)\right)\\ &= \frac{d^2y}{dt^2} - (r_1 + r_2)\frac{dy}{dt} + r_1 r_2 y(t).\end{aligned}$$

This composition can be thought of as a product[1]. One thing we notice is that the composition gives rise to a second order differential operator. In fact, from the above we have

$$\left(\frac{d}{dt} - r_1\right)\left(\frac{d}{dt} - r_2\right) = \frac{d^2}{dt^2} - (r_1 + r_2)\frac{d}{dt} + r_1 r_2. \tag{1}$$

On the other hand, given a second order differential operator coefficient, say

$$\frac{d^2}{dt^2} + b\frac{d}{dt} + c,$$

one may ask whether it is possible to factorize it so that it becomes the product of two first order differential operators. The answer is already given in formula (1) above; if r_1, r_2 are the two roots of the quadratic equation

$$r^2 + br + c = 0,$$

[1] In fact this is the point of view one would take if one studies group theory.

then $r_1 + r_2 = -b$ and $r_1 r_2 = c$, so by (1),

$$\frac{d^2}{dt^2} + b\frac{d}{dt} + c = \left(\frac{d}{dt} - r_1\right)\left(\frac{d}{dt} - r_2\right).$$

This formula is very useful in solving second order ODEs with constant coefficients. For example, if one wants to solve

$$y''(t) + by'(t) + cy(t) = 0, \tag{2}$$

where b, c are constants, then one can first write this equation as

$$\left(\frac{d^2}{dt^2} + b\frac{d}{dt} + c\right) y(t) = 0. \tag{3}$$

Now the differential operator on the left hand side can be factorized as a product of two first order differential operators: if r_1, r_2 are the two roots of the quadratic equation

$$r^2 + br + c = 0,$$

then the ODE (3) becomes

$$\left(\frac{d}{dt} - r_1\right)\left(\frac{d}{dt} - r_2\right) y(t) = 0.$$

Let

$$v(t) = \left(\frac{d}{dt} - r_2\right) y(t). \tag{4}$$

Then from the above, we have

$$\left(\frac{d}{dt} - r_1\right) v(t) = 0. \tag{5}$$

But this is a first order equation in $v(t)$, which can be solved explicitly. Once $v(t)$ is found, it can be plugged back into (4), and one can then solve for $y(t)$.

In fact, by solving (5), it is easy to check that $v(t)$ is given by

$$v(t) = C_1 e^{r_1 t}$$

for some constant C_1. Thus (4) becomes

$$\left(\frac{d}{dt} - r_2\right) y(t) = C_1 e^{r_1 t},$$

or

$$y'(t) - r_2 y(t) = C_1 e^{r_1 t}.$$

This is a first order linear equation, which one can solve using integrating factors: in fact the integrating factor is $e^{-r_2 t}$, so we have

$$e^{-r_2 t} y' - r_2 e^{-r_2 t} y = C_1 e^{(r_1 - r_2)t},$$

or

$$(e^{-r_2 t} y)' = C_1 e^{(r_1 - r_2)t}. \tag{6}$$

Now there are two cases: either $r_1 \neq r_2$, or $r_1 = r_2$. We consider them separately:

Case 1: $r_1 \neq r_2$ Then

$$e^{-r_2 t} y = \frac{C_1}{r_1 - r_2} e^{(r_1 - r_2)t} + C_2,$$

for some constant C_2, so

$$y(t) = \frac{C_1}{r_1 - r_2} e^{r_1 t} + C_2 e^{r_2 t},$$

and by renaming the constants we get

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

for some constants C_1, C_2. (Note that we used $r_1 \neq r_2$ to integrate $e^{(r_1 - r_2)t}$!)

Case 2: $r_1 = r_2$ Then (6) becomes,

$$(e^{-r_2 t} y)' = C_1,$$

so

$$e^{-r_2 t} y = C_1 t + C_2,$$

for some constant C_2, i.e.

$$y(t) = C_1 t e^{r_2 t} + C_2 e^{r_2 t}.$$

This provides a rigorous justification of the solution formula for the ODE (2) we saw in class.