Math 252 Spring 2012

Review for the final exam

1. Solve the initial value problem

$$y'(t) + 2y(t) = 4te^{-2t}, \quad y(0) = 1.$$

2. Find all solutions of the equation $(1+t^2)\sec^2(y)\dfrac{dy}{dt} = 2t$.

3. For the equation

$$y'(t) = \sin(y(t)) :$$

   (a) Find all equilibrium solutions; (Hint: There are infinitely many.)

   (b) Draw (a representative portion of) the phase line, and classify the equilibrium solutions as sources or sinks;

   (c) Draw the slope field of the equation on the y-t plane;

   (d) Draw some typical solutions of the equation on the y-t plane.

4. (a) Draw the bifurcation diagram for the one-parameter family of equations

$$y' = y^2 - \mu y,$$

   where μ is a real parameter. Where does bifurcation happen?

   (b) Draw the bifurcation diagram for the one-parameter family of equations

$$y' = \sin(y) + \mu,$$

   where μ is a real parameter. Where does bifurcation happen?

5. For each of the following equations, find all real-valued functions $y(t)$ that satisfies that:

   (a) $y''(t) + 4y'(t) + 3y(t) = 0$;

   (b) $y''(t) + 4y'(t) + 4y(t) = 0$;

   (c) $y''(t) + 4y'(t) + 5y(t) = 0$.

   What is the point of asking you to solve all three equations? (Hint: The solution formula are different in each case!)

6. Describe the qualitative long-time behavior of the genearal solution of the equation

$$y'' + 6y' + \mu y = 0,$$

where μ is a positive parameter. You should sketch the graphs of solutions for some typical values of μ, and find all values of μ where there is a qualitative change in the behavior of the solution.

7. Relate what you have written down in the previous question to the study of damped harmonic oscillators.
8. Solve the initial value problem

$$\begin{cases} y''(t) + 4y'(t) + 3y(t) = e^{-5t} \\ y(0) = -1, \quad y'(0) = 2 \end{cases}.$$

What difference do you expect, if the right hand side of the ordinary differential equation is e^{-3t} instead?

9. Solve the initial value problem

$$\begin{cases} y''(t) + 9y(t) = 4\cos(3t) \\ y(0) = 0, \quad y'(0) = 0 \end{cases}.$$

Describe the behavior of the solution as $t \to \infty$, and relate this behavior to resonance.

10. Find the general solution of the equation

$$y''(t) + 2y'(t) + 5y(t) = 4\sin(t).$$

Sometimes it is said that this equation has a steady state solution. Explain what that means, and rewrite the steady state solution in the form

$$C\cos(\omega t + \theta)$$

for some constants C, ω and θ.

11. For each the following systems of equation,
    (a) Rewrite it in matrix form;
    (b) Find all real-valued solutions;
    (c) Find all equilibrium solutions;
    (d) Classify the origin as a source / sink / saddle / spiral source / spiral sink / center (if appropriate);
    (e) Sketch the phase portrait. You should draw all the equilibrium solutions and straight line solutions, as well as some representative curved solutions of the system. You should also put arrows on the non-equilibrium solutions

you draw. Should there be any asymptotics as $t \to \pm\infty$, you should pay attention to those as well. For spiral sources / spiral sinks / centers, you should be careful about whether the solution goes around the origin in a clockwise / counter-clockwise manner.

(i) $\begin{cases} x' = 2x + y \\ y' = -3x - y \end{cases}$

(ii) $\begin{cases} x' = -2x + y \\ y' = x - 2y \end{cases}$

(iii) $\begin{cases} x' = 3x - 6y \\ y' = 2x - 4y \end{cases}$

(iv) $\begin{cases} x' = 3x - 7y \\ y' = -2y \end{cases}$

(v) $\begin{cases} x' = x + 4y \\ y' = -x + 5y \end{cases}$

(vi) $\begin{cases} x' = -2x + y \\ y' = -9x - 2y \end{cases}$

(vii) $\begin{cases} x' = 3x + y \\ y' = 2x + 2y \end{cases}$

12. Solve the following initial value problems:

(a) $\begin{cases} x' = x + 4y \\ y' = -x + 5y \end{cases}, \quad \begin{cases} x(0) = 1 \\ y(0) = -2 \end{cases}$

(b) $\begin{cases} x' = -2x + y \\ y' = -9x - 2y \end{cases}, \quad \begin{cases} x(0) = 1 \\ y(0) = 0 \end{cases}$

(c) $\begin{cases} x' = 3x + y \\ y' = 2x + 2y \end{cases}$, $\begin{cases} x(0) = 3 \\ y(0) = -3 \end{cases}$

(Hint: The systems in parts (a), (b), (c) are the systems in (v), (vi) and (vii) of the previous question.)

13. If A is the coefficient matrix of the system in part (c) of the previous question, compute e^{At} for any real number t. Use this to solve part (c) of the previous question again.
14. Given a second order equation

$$y'' + by' + cy = 0$$

with constant coefficients b and c,

(a) Rewrite this second order equation as a system;

(b) If A is the coefficient matrix of the system you wrote down in part (a), write down the quadratic equation that one would solve to find the eigenvalues of A. Does this quadratic equation look familiar?

15. The following system could be a model for a predator-prey system:

$$\begin{cases} x' = 2x\left(1 - \frac{x}{2}\right) - xy \\ y' = -2y + 2xy \end{cases}$$

(a) Find all equilibrium solutions of the system. (Answer: There are three of them, namely (0,0), (2,0) and (1,1).)

(b) Find the linearization of the system around the equilibrium point $(0, 0)$. Classify it as a source / sink / saddle / spiral source / spiral sink.

(c) Find the linearization of the system around the equilibrium point $(1, 1)$. Classify it as a source / sink / saddle / spiral source / spiral sink.

(d) According to your computation above, what do you think will happen if at time $t = 0$, the population of the species is $(1.1, 0.9)$? Explain.

(e) Confirm your answer in the previous parts by drawing the phase portrait of the system on a computer.

16. (a) Compute the Laplace transform of the functions e^{-2t}, te^{-2t} and t^2e^{-2t};

(b) Use this to solve Question 1 again.