Math 252 Spring 2012

Homework 1

You should solve these problems without using a calculator.

1. Find the derivatives of the following functions of t (wherever they are defined).

$$\cos(2t), \qquad te^{-t^2}, \qquad \sec^2(3t), \qquad \frac{t}{1+9t^2}, \qquad \frac{1}{\sqrt{1-4t^2}}$$

2. Find the anti-derivatives of the functions in Question 1.

3. Find all functions $y(t)$ satisfying

$$\frac{dy}{dt} = \cos^3(2t) + t\sin t.$$

4. Find all functions $y(t)$, defined for $t > 0$, such that

$$\frac{dy}{dt} = t\log t, \quad \text{and} \quad y(1) = 3.$$

5. (a) Sketch a graph of $y(t) = \sin(2t)$ for $0 \leq t \leq 2\pi$.
   (b) Where is $y(t)$ increasing within the above range of t?
   (c) Repeat the above if now $y(t) = t\sin(2t)$ instead.

6. Let A be the following matrix and b be the following column vector:

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

   (a) Find all column vectors x such that $Ax + b = 0$.
   (b) Find all eigenvalues of A. For each eigenvalue, find all the eigenvectors of A corresponding to that eigenvalue.
   (c) How many eigenvalues of A are positive, and how many are negative?
   (d) Repeat the above if now

$$A = \begin{pmatrix} 7 & 9 \\ -4 & -5 \end{pmatrix}$$

   instead.

7. Find the Jacobian matrix of the map

$$F(x, y) = (-3\sin x + y, 4x + 3\cos y - 3)$$

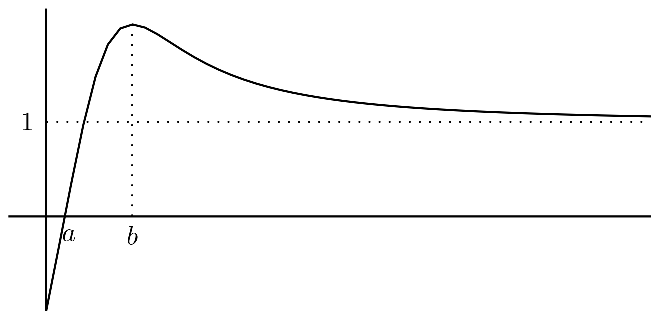
   at the point (x, y). Find the eigenvalues of this matrix when $(x, y) = (0, 0)$.

8. Find the eigenvalues of the Jacobian matrix of the map

$$F(x, y) = (2x - x^2 - xy, 3y - y^2 - 2xy)$$

at the point $(1, 1)$. How many of them are positive? negative? zero?

9. In the following figure, the horizontal axis is the t-axis, and the vertical axis is the y-axis. The solid curve is the graph of $y = f(t)$ for some function $f(t)$, defined for $t \geq 0$.



Based on what is suggested in the graph:

(a) Decide the range of t where $f(t)$ is positive.

(b) Decide the range of t where $f'(t)$ is positive.

(c) Predict whether

$$\lim_{t \to +\infty} f(t) \quad \text{and} \lim_{t \to +\infty} f'(t)$$

exist, and what their values are if they exist.

10. Compute the following limits:

$$\lim_{t \to \infty} \frac{1 + 5t - 4t^2}{1 + 3t^2}, \qquad \lim_{t \to \infty} \frac{\sin t}{t}, \qquad \lim_{t \to \infty} \frac{\log(1 + t^2)}{t(t + 1)}.$$

11. For each of the following statements, decide whether it is true or false. Give a counter-example if the statement is false.

(a) If $f(t) \geq 0$ for all $a < t < b$, then $f(t)$ is increasing on the interval (a, b).

(b) If $f'(t) \geq 0$ for all $a < t < b$, then $f(t)$ is increasing on the interval (a, b).

(c) If $f''(t) \geq 0$ for all $a < t < b$, then $f(t)$ is increasing on the interval (a, b).

12. (a) Show that the vectors $(1, 3)$ and $(2, -4)$ are linearly independent over $\mathbb{R}$.

(b) Give an example of two vectors in $\mathbb{R}^2$ that are not linearly independent over $\mathbb{R}$.

(c) (Optional) Show that the functions $\sin(t)$ and $\cos(t)$ are linearly independent over $\mathbb{R}$.